Edge-connectivity augmentation of graphs and hypergraphs

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Contents

Chapter 1

Introduction

1.1 Acknowledgements - Köszönetnyilvánítás

Köszönetet szeretnék mondani mindenekelőtt a családomnak. Köszönöm a szüleimnek a sok bíztatást és lehetőséget: megtanultam tőlük a munka szeretetét és azt, hogy a legjobb motiváció a dícséret. Köszönöm feleségemnek, hogy a nehéz pillanatokban se hagyott elcsüggedni. Köszönöm a gyermekeimnek, hogy megtanítottak értékelni és kihasználni azt az időt amit együtt tölthetünk, és nemkülömben azt az időt, amit külön töltünk.

Köszönöm pedagógusaimnak, tanáraimnak, akik az általános iskolától, sőt az óvodától kezdve bontogatták az érdeklődésemet a világ, azon belül is a matematika iránt. Hadd említsem nevükön a matematika tanáraimat. Köszönöm Molnár Éva 1-2. osztályos tanító nénimnek, Tóth János 3-4. osztályos tanító bácsimnak, Németh Ágnes felső tagozatos tanár nénimnek az általános iskolából, Bíróné Mihályfi Erzsébet tanárnőnek a gimnáziumból.

Az egyetem matematika-fizika, később matematikus szakán már sok matematika tanár nevét kellene megemlítenem, ezért nem teszek kísérletet egy teljes felsorolásra. Kiemelném viszont Szamuely Tamás algebra gyakorlatvezetőmet, aki először bíztatott arra, hogy matematikus szakra jöjjek. Köszönöm továbbá Frank Andrásnak, hogy lebilincselő előadásaival olthatatlan tudásszomjat ébresztett bennem a kombinatorikus optimalizálás és a gráfelmélet iránt, és hogy rengeteg szakmai segítséget nyújtott és anyagi lehetőséget biztosított arra, hogy kutassam ezeket a területeket.

Köszönöm témavezetőmnek, Vizvári Bélának, hogy elvállalta a témavezetést. Köszönöm továbbá kollégáimnak a sok együttműködést és segítséget. Közülük is elősorban konzulensemnek, Király Tamásnak volt meghatározó szerepe abban, hogy ez a dolgozat egyáltalán megszületett, mert mindig hajlandó volt végighallgatni az elképzeléseimet, mindig számítani lehetett rá, hogy észreveszi (és általában ki is javítja) a hibákat a gondolatmenetemben. Köszönöm a társszerzőimnek, hogy megpróbáltak velem együtt gondolkodni, megosz-

tották velem részeredményeiket és ötleteiket és meghallgatták az enyémeket. Hadd említsem őket ismét név szerint, még ha néhányukkal a közös publikáció nem is szerepel ebben a disszertációban: Henning Bruhn, Gerbner Dániel, Roland Grappe, Satoru Iwata, Gwenaël Joret, Király Zoltán, Szigeti Zoltán, és természetesen Király Tamás. Köszönöm továbbá az EGRES csoport minden tagjának a sok segítséget, hogy elviseltek a disszertáció előkészítésének és megírásának minden (különösen a legvégső) periódusában, ellenőrizték a kézirataimat és meghallgatták a bizonyításvázlataimat.

1.2 Overview

This thesis is devoted to **edge-connectivity augmentation** which is the following: we are given a network (modelled by a graph or hypergraph) that we somehow want to make more robust against the failure (deletion) of its connections (edges or hyperedges). The operation that we can do is to **introduce new connections between already existing nodes**. Most of this thesis is about this notion of edge-connectivity augmentation (apart from the last chapter about source location where we consider a different augmentation technique – more details later). We emphasize that though we sometimes allow directed structures as the basic network to be augmented (directed graphs and hypergraphs, or more generally mixed graphs and hypergraphs), **the new connections to be added will always be undirected**. This is because our techniques (e.g. polyhedral considerations) do not apply when we augment with directed arcs or hyperarcs.

The objective function of the augmentation is also common to the problems we investigate: we want to **minimize the total size of the hyperedges** to be added. When we are only allowed to add graph edges then this is two times the number of the new edges. A little bit more general objective function is also considered in the so called **minimum node-cost augmentation problems** (exact definition to be given later). The reason we consider these objective functions is simple: the polyhedral relations show that these are tractable, while if a cost of a new hyperedge does not only depend on the cost of its vertices then the simplest augmentation problems already become intractable.

The starting point of our investigations is Menger's theorem (with its many different versions), which allows us to reformulate the augmentation problem into a **covering problem**. By covering problem we mean a problem of the following form: given a set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ over the finite ground set V, find a graph (or hypergraph) $G = (V, E)$ **covering** the function p, meaning that $d_G(X) \geq p(X)$ has to hold for any subset $X \subseteq V$ (here $d_G(X)$ denotes the number of (hyper)edges of G intersecting both X and $V - X$). A whole chapter (Chapter 2) is devoted to showing how different versions of the edge-connectivity augmentation problem can be formulated as a covering problem with a suitably chosen **requirement (or deficiency) function** p.

The above notion of edge-connectivity augmentation is quite well studied and has a huge literature. For two recent surveys we refer the reader to [43] and [21]. Let us first (in this and the next paragraph) restrict ourselves to **augmentation with graph edges**. The first result is due to Watanabe and Nakamura [45] on *global edge-connectivity augmentation of graphs*. An important milestone is due to Cai and Sun [13] and Frank [19] who realized that the splitting-off technique introduced by Lovász [33] is a very useful tool in solving edge-connectivity augmentation problems with graph edges. This method will also be used in this thesis. Frank observed the importance of skew-supermodular functions and proved in [19] the deep result that is behind the connection between degree-specified augmentation and augmentation with a minimum number of edges. With this method and the splitting-off theorem due to Mader [34], Frank [19] solved the *local edge-connectivity augmentation of graphs*. Another result due to Bang-Jensen, Frank and Jackson [2] uses a similar approach to *increase the global arc-connectivity of mixed graphs with undirected edges*. A recent approach is the *node-to-area connectivity augmentation* introduced by Japanese researchers. Here the requirement is not between node-pairs, but between nodes and nodesets. Though the basic problem is NP-hard, a surprising observation due to Ishii and Hagiwara [26] says that if we restrict ourselves to the (most interesting) case when the requirements are all at least 2, then the problem becomes tractable. Bang-Jensen and Jackson [4] generalized the result of Watanabe and Nakamura in an other direction and solved the problem of *augmenting the global edge-connectivity of a hypergraph with graph edges*. An abstract version of this result, the problem of *covering a symmetric crossing supermodular function with graph edges* was solved by Benczúr and Frank [5]. Yet another generalization of Watanabe and Nakamura's result, the *partition constrained global edge-connectivity augmentation of graphs* was solved by Bang-Jensen, Gabow, Jordán and Szigeti [3].

What can we add to the widely studied area of edge-connectivity augmentation (with graph edges) in this thesis? We have found a new and convenient approach to the splitting-off technique which simplifies the discussion to a great extent. The main breakthrough is a simple lemma (Lemma 4.6) that, instead of only looking at dangerous sets (that are the important objects in splitting-off theorems) as in previous proofs, considers the interrelation between dangerous sets and sets with maximum deficiency and thus simpifies the further discussion. This gives a simplification of many of the known proofs and enables us to prove new results, too. Furthermore, we have given an unified discussion of the results mentioned above based on the properties of the deficiency functions of the problems. The most general class of set functions is skew-supermodular functions that are hard to handle in general (the symmetry of our function can be assumed by the **symmetrizing operation**). However, in many cases the function to be handled is from a more restricted class, for example it is the symmetrized of a crossing supermodular function as in global arc-connectivity augmentation of mixed hypergraphs, or it is the symmetrized of a crossing negamodular function as in the node-to-area connectivity augmentation problem.

If we are also allowed to use **hyperedges of arbitrary size** then the problems become simpler. Our starting point is Theorem 3.2 of Szigeti on covering skew-supermodular functions with hyperedges. The proof of this theorem was considerably simplified by Tamás Király, who used the hyperedge merging technique, which might be considered as a generalization of splitting-off. This proof is very fascinating, and with some further observations and some polyhedral results we (Tamás Király and the author) managed to generalize Szigeti's theorem in many directions. For example we have shown that the hypergraph covering our function can be chosen **nearly uniform**, or in some restricted cases we can even solve the **simultaneous covering of two functions** optimally. Applications include the local edge-connectivity augmentation of hypergraphs, the global arc-conncetivity of mixed hypergraphs with undirected hyperedges, and the node-to-area connectivity augmentation in hypergraphs.

In Chapter 6 we consider versions of the **source location problem**, which can be interpreted as a different notion of edge-connectivity augmentation. Here, instead of adding new (hyper)edges connecting existsing nodes, we are allowed to **contract** a suitably chosen set of nodes (called **source set**). The motivation of this problem comes from the following application: given a network consisiting of computers, say, and some connections between them, we want to decide which of these computers to use as servers for a service that we want to provide for our users (who are also located at the computers of this networks). The requirement is that the users must have good edge-connectivity to the set of servers, and the cost of choosing a node as a server may vary from one node to the other. We consider hypergraphic generalizations of certain source location problems in Chapter 6.

In this thesis we will only concentrate on polynomial solvability of the problems. We will often explicitly describe algorithms, too, and we show that they have a polynomial running time, but usually we do not aim to achieve the best running times. Instead, we try to find a conceptually simple algorithm for our problem.

The **structure of the thesis** is as follows. In the rest of this chapter we introduce the most important notations and definitions that we will need, we introduce the appropriate oracles needed in our abstract set covering algorithms, and we recall the important results about g-polymatroids.

Chapter 2 is still an introductory chapter. Here we first of all show how the edge-connectivity augmentation problems that we plan to speak about can be reformulated to covering problems. This means the introduction of deficiency functions and showing their main properties. This can be found in Section 2.1. In Section 2.2 we show the tractable objective functions of our optimization problems, and we show the connection between degree-specified covering and the minimum version. Lastly, in Section 2.3 we recall the basic facts about the splitting-off technique needed later.

Chapter 3 is devoted to covering a skew-supermodular function with hyperedges. First we give the result on merging hyperedges originally due to Tamás Király, but in a generalized form. In Section 3.2 we introduce the notion of **weak covering of a set function** and we show how this relates to g-polymatroids. Furthermore we show that if we use the smallest number of hyperedges then weak covering implies covering. These are the main building blocks of our results on covering a skew-supermodular function with nearly uniform hyperedges and covering two skew-supermodular functions simultaneously. In Section 3.4 we state the applications of these results.

In Chapter 4 we turn to covering skew-supermodular functions with graph edges. After a brief review of previous results we give our key lemma, Lemma 4.6. In fact this lemma was also observed by Nutov in 2005, but he did not publish a complete proof of it (a special case of this lemma also appears in the Ph.D. thesis of Ben Cosh [15], but the proof is rather cumbersome). We have found a very simple proof for this lemma. We look at the consequences of this lemma in the subsequent sections: we show how a greedy type splitting-off algorithm would get stuck in general (Section 4.2.1), and why it will not get stuck in many well known special cases (Section 4.2.2 on simple proofs of known results). Then in Section 4.3 we describe (as far as we can) this stuck situation for special skew-supermodular functions. In Section 4.4 we give applications of our results of the previous sections.

Chapter 5 is devoted to covering symmetric crossing supermodular functions with graph edges. This is a special case of the problem of the previous chapter, however we thought that this order moving "from the general to the special" is more convenient in our discussion. After reviewing the known results and giving some preliminaries, we give a relatively simple proof of the theorem of Benczúr and Frank that solves the problem of covering a symmetric crossing supermodular function with graph edges. This proof enables us to generalize the result further by posing extra conditions on the graph edges allowed in the covering. The problem considered is the partition constrained version of covering a symmetric crossing supermodular function. We give a solution of this problem in Section 5.4. In Section 5.4.4 we describe the application of global edge-connectivity augmentation of a hypergraph with a multipartite graph.

Chapter 6 is devoted to hypergraphic versions of the source location problem. In Section 6.1 we introduce the problems that we want to solve: in particular we introduce an abstract version of the source location problem. In Section 6.2 we give a greedy type algorithm that solves the tractable version (that is, compatible requirement- and weight function) of our problem in general. However, it is not clear how to implement a step of it, since it is not known how to minimize an intersecting posimodular function in polynomial time. In Section 6.3 we show how to implement this step of the previous algorithm if the function is also submodular, which is true for the hypergraphic source location. Finally, in Section 6.4 we show a cleverer and faster algorithm for the case when the requirement function is uniform.

In Chapter 7 we enumerate some open problems that, if solved, would have fit into this thesis.

1.3 Notations and preliminaries

In the following sections we introduce the notions and notations that will be used in the thesis.

1.3.1 Graphs and hypergraphs

In the whole thesis V will be a finite ground set and we will usually use $n = |V|$. For subsets X and Y of V, we use the notation $X - Y := \{v \in X : v \notin Y\}$, $X + Y = X \cup Y$, and $\overline{X} = V - X$ (if the ground set is clear from the context). Two subsets $X, Y \subseteq V$ are called **co-disjoint** if $X \cup Y = V$.

A **partition** of the set V is a collection $X_1, X_2, \ldots, X_t$ of nonempty disjoint subsets of V with $V = \cup_1^t X_i$ (in some cases we allow that some members of a partition can be empty, but we will always state this explicitly). A **subpartition** of V is a partition of a subset of V. Let $\mathcal{S}(V)$ denote the set of all subpartitions of the set V. If a set X has only one element x then we will call it a **singleton** and we will often write x instead of $\{x\}$ (for example if $d : 2^V \to \mathbb{R}$ is a set function then $d(x)$ means $d(\{x\})$). The characteristic function of a set X will be denoted by $\chi_X : V \to \{0;1\}$, i.e. $\chi_X(v) = 1$ if $v \in X$ and $\chi_X(v) = 0$ otherwise. For $s, t \in V$, a set $X \subseteq V$ is an $\overline{s}t$**-set** if $s \notin X$ and $t \in X$.

In the thesis $\mathbb{Z}_+$ ($\mathbb{Q}_+$, $\mathbb{R}_+$) will denote the set of non-negative integer (rational, real) numbers.

Sets $X, Y \subseteq V$ are **properly intersecting** if $X \cap Y, X - Y$ and $Y - X$ are all nonempty: this terminology is due to András Frank and we decided to use it in order to clarify the distinction between intersecting (i.e. not disjoint) set pairs and properly intersecting set pairs. If furthermore $X \cup Y \neq V$ then we say that the two sets are **crossing**. For a family $\mathcal{F} \subseteq 2^V$ let $\mathrm{co}(\mathcal{F}) = \{X \subseteq V : \overline{X} \in \mathcal{F}\}$. We say that $\mathcal{F}$ is a **ring family** if $X \cap Y, X \cup Y \in \mathcal{F}$ holds for any $X, Y \in \mathcal{F}$. We say that $\mathcal{F}$ is an **intersecting family** (**crossing family**) if $X \cap Y, X \cup Y \in \mathcal{F}$ holds for any intersecting (respectively crossing) pair $X, Y \in \mathcal{F}$.

An **undirected hypergraph** (or shortly **hypergraph**) $H = (V, \mathcal{E})$ is a pair of a finite set V and a family $\mathcal{E}$ of subsets of V (repetitions are allowed). The set V is called the **node set** of the hypergraph, the family $\mathcal{E}$ is called the **edge set** of the hypergraph. An element e of $\mathcal{E}$ will be called a **hyperedge**. The cardinality of e is denoted by $|e|$.

If the cardinality of every hyperedge is 2 in a hypergraph then it is called a **graph**. A graph is usually denoted by $G = (V, E)$, the elements of E are called **edges**. A graph can contain parallel edges and uv or (u, v) will denote an arbitrary edge between the nodes $u, v \in V$. Note however that in some cases we will allow **loops** in graphs, so strictly speaking a graph (with loops) is not a special case of a hypergraph, but this small abuse of notation will not cause any misunderstanding.

In a hypergraph H, a **path** between nodes s and t is an alternating sequence of distinct nodes and hyperedges $s = v_0, e_1, v_1, e_2, \ldots, e_k, v_k = t$, such that $v_{i-1}, v_i \in e_i$ for all i between 1 and k. H is **connected** if there is a path between any two distinct nodes.

For a hypergraph $H = (V, \mathcal{E})$ and a set $X \subseteq V$ we say that a hyperedge $e \in \mathcal{E}$ enters X if neither $e \cap X$ nor $e \cap (V - X)$ is empty, and we define $\Delta_H(X) = \{e \in \mathcal{E} : e \text{ enters } X\}$ (the set of hyperedges entering X) and $d_H(X) = \Delta_H(X)$ (the **degree** of X in H). The set function $d_H\colon 2^V \to \mathbb{Z}_+$ is a symmetric submodular function (see the definition in Section 1.3.2). For graphs we will sometimes have to count loop edges in the degree of singetons, therefore we introduce the function $d_G^+ : V \to \mathbb{Z}$ for a graph $G = (V, E)$ to mean that $d_G^+(v)$ is $d_G(v)$ plus two times the number of loops incident to v (for a hypergraph H let $d_H^+(v)$ be $d_H(v)$ plus the number of singleton hyperedges $\{v\}$). For a graph $G = (V, E)$ and $X, Y \subseteq V$ we introduce the notation $d_G(X, Y)$ to be equal to the number of edges of G with one endpoint in $X - Y$ and the other in $Y - X$. Let $\overline{d}_G(X, Y) = d_G(X, \overline{Y}) = d_G(Y, \overline{X})$. We will use that the following equalities hold for any graph $G = (V, E)$ and $X, Y \subseteq V$.

$$d_G(X) + d_G(Y) = d_G(X \cap Y) + d_G(X \cup Y) + 2d_G(X, Y), \tag{1.1}$$

$$d_G(X) + d_G(Y) = d_G(X - Y) + d_G(Y - X) + 2\overline{d}_G(X, Y). \tag{1.2}$$

More generally, one can prove the following equalities for any hypergraph $H = (V, \mathcal{E})$ and $X, Y \subseteq V$.

$$d_H(X) + d_H(Y) = d_H(X \cup Y) + d_H(X \cap Y) + 2d_1(X, Y) + d_2(X, Y), \tag{1.3}$$

$$d_H(X) + d_H(Y) = d_H(Y - X) + d_H(X - Y) + 2\overline{d_1}(X, Y) + \overline{d_2}(X, Y), \tag{1.4}$$

where $\boldsymbol{d_1(X,Y)}$ is the number of hyperedges intersecting only $X-Y$ and $Y-X$ and neither $X\cap Y$ nor $V-(X\cup Y)$, $\boldsymbol{d_2(X,Y)}$ is the number of hyperedges intersecting $X-Y$ and $Y-X$ and exactly one of $X\cap Y$ and $V-(X\cup Y)$, and $\overline{d_1}(X,Y)=d_1(X,\overline{Y})=d_1(\overline{X},Y)$ and $\overline{d_2}(X,Y)=d_2(X,\overline{Y})=d_2(\overline{X},Y)$ (i.e. $\boldsymbol{\overline{d_1}(X,Y)}$ is the number of hyperedges intersecting only $X\cap Y$ and $V-(X\cup Y)$, $\boldsymbol{\overline{d_2}(X,Y)}$ the number of hyperedges intersecting $X\cap Y$ and $V-(X\cup Y)$ and exactly one of $X-Y$ and $Y-X$).

We say that the **hypergraph** H **covers a set function** p if $d_H(X)\geq p(X)$ for any $X\subseteq V$ (shortly $d_H\geq p$). The **total size of the hypergraph** is the sum of the cardinalities of the hyperedges: if our hypergraph is a graph then this is two times the number of the edges of this graph. The **rank of a hypergraph** is the size of the largest hyperedge in it. A hypergraph is said to be **uniform** if the size of every hyperedge is the same. We say that the hypergraph is **nearly uniform**, if the largest hyperedge is at most one bigger than the smallest one.

Since the thesis is about edge-connectivity augmentation, the following definition is of basic importance.

Definition 1.1. *Given a hypergraph $H=(V,\mathcal{E})$ and sets $X,Y\subseteq V$, the* ***edge-connectivity between*** *X* ***and*** *Y, denoted by $\lambda_H(X,Y)$, is the maximum number of edge-disjoint paths starting in X and ending in Y (we say that $\lambda_H(X,Y)=\infty$ if $X\cap Y\neq\emptyset$). The subscript H may be omitted if no confusion can arise.*

It is well known that Menger's theorem can be generalized for hypergraphs:

Theorem 1.2. *Let $H=(V,\mathcal{E})$ be a hypergraph, and $S,T\subseteq V$. Then*

$$\lambda_H(S,T)=\min\{d_H(X): T\subseteq X\subseteq V-S\}.$$

For nodes $u,v\in V$ we will also say that $\lambda_H(u,v)$ is the **local edge-connectivity between** u **and** v. A hypergraph $H=(V,\mathcal{E})$ is k**-edge-connected** if $\lambda_H(u,v)\geq k$ for any $u,v\in V$ (where k is a positive integer). By Menger's theorem this is equivalent to saying that $d_H(X)\geq k$ for any nonempty $X\subsetneq V$. More generally, for a function $r\colon V\times V\to\mathbb{Z}_+$ we say that H **is** r**-edge-connected** if $\lambda_H(u,v)\geq r(u,v)$ for any $u,v\in V$. Observe that we can assume that r is symmetric, i.e. $r(u,v)=r(v,u)$ for any $u,v\in V$. This follows from the observation that H is r-edge-connected if and only if H is r'-edge-connected, where $r'(u,v)=\max\{r(u,v),r(v,u)\}$ for any $u,v\in V$.

In some cases it will be useful to allow in undirected hypergraphs that nodes have multiplicities in the hyperedges. So let us introduce the notion of **multihypergraph** as a pair $H=(V,\mathcal{E})$ of a finite set V and a family $\mathcal{E}=\{e\colon V\to\mathbb{Z}_+\}$ of functions, called **multihyperedges** (sometimes simply hyperedges). Multihyperedges are considered to be multisets identified by their characteristic functions, i.e. $e(v)$ equals the multiplicity of the node v in the hyperedge e. This is just a natural generalization of loop edges in graphs. A hypergraph is clearly a special case of a multihypergraph, if we identify the hyperedges with their characteristic vectors. For a multihypergraph we can define the **underlying hypergraph** that simply means that we forget about the multiplicities of the nodes in the hyperedges (i.e. if $H = (V, \mathcal{E})$ is a multihypergraph then the underlying hypergraph of H is $(V, \{\max(e, 1) : e \in \mathcal{E}\})$). Most of the definitions for multihypergraphs do not use the multiplicities of the nodes in the multihyperedges: if $H = (V, \mathcal{E})$ is a multihypergraph then **a path between two nodes**, the **local edge-connectivity** in H, the **degree of a set** $X \subseteq V$ **is defined through the underlying hypergraph** of H. The most important difference is in the definition of the function d^+_H: for a multihypergraph $H = (V, \mathcal{E})$ and node $v \in V$, $d^+_H(v) = \sum_{e \in \mathcal{E}} e(v)$. Also, the cardinality of a multihyperedge e is defined with $|e| = \sum_{v \in V} e(v)$. A function $m : v \to \mathbb{Z}_+$ will often be called a **degree specification**. We say that the multihypergraph H **satisfies the degree-specification** $m : v \to \mathbb{Z}_+$ if $d^+_H(v) = m(v)$ for every $v \in V$. We note that the notion of multihypergraphs is not very important in this thesis: we only introduce them because some theorems have a simpler form if we formulate them for multihypergraphs instead of hypergraphs, and we will try to show these simplifications. In fact we only speak about multihypergraphs in Chapter 3.

Another generalization of a hypergraph is a mixed hypergraph. A **mixed hypergraph** $M = (V, \mathcal{A})$ is a pair of a finite set V and a family $\mathcal{A}$ containing nonempty ordered pairs of subsets of V (the same pair can occur more than once). The elements of $\mathcal{A}$ are called **hyperarcs**. For a hyperarc $a = (T_a, H_a) \in \mathcal{A}$, the set T_a is called the **tail set** of a, while H_a is called the **head set** of a. More intuitively we can think of a hyperarc a as the subset $T_a \cup H_a$ of V, in which every node is either a **head node**, a **tail node** or even both (**head-tail node**), such that every hyperarc contains at least one head and at least one tail. Hopefully it will not cause any confusion if we sometimes refer to a hyperarc $a = (T_a, H_a)$ as a single set $T_a \cup H_a$: for example this way we define **the size of** a as $|a| = |T_a \cup H_a|$, or the **rank of a mixed hypergraph** as the maximum size of its hyperarcs. When we say that v is a tail node of a hyperarc a then we also allow that it is a head-tail node (and similarly for head nodes). An undirected hypergraph can be considered (for our purposes) as a special mixed hypergraph where every node in a hyperarc is a head-tail node of this hyperarc. For a mixed hypergraph $M = (V, \mathcal{A})$ and some $X \subseteq V$, the restriction of M to X is a mixed hypergraph defined as $M[X] = (X, \{a \in \mathcal{A} : a \subseteq X\})$.

A **mixed graph** is the special case of a mixed hypergraph where every hyperarc is of cardinality two. For a mixed graph G and sets $X, Y \subseteq V$ let $d_G(X, Y)$ denote the number of (undirected or directed) edges of G with one endpoint in $X - Y$ and the other in $Y - X$.

In a mixed hypergraph M, a **path** between nodes s and t is an alternating sequence of distinct nodes and hyperarcs $s = v_0, a_1, v_1, a_2, \dots, a_k, v_k = t$, such that v_{i-1} is a tail node of a_i and v_i is a head node of a_i for all i between 1 and k. A hyperarc a **enters a set** X if there is a head node of a in X and there is a tail node of a in $V - X$. For a set X we define $\varrho_M(X) = |\{a \in \mathcal{A} : a \text{ enters } X\}|$ (the **in-degree** of X) and $\delta_M(X) = \varrho_M(V - X)$ (the **out-degree** of X). It is easy to check that both ϱ and δ are submodular (again, see the definition later). Given a mixed hypergraph $M = (V, \mathcal{A})$ and sets $S, T \subseteq V$, let $\lambda_M(S, T)$ denote the maximum number of arc-disjoint paths starting in S and ending in T (we say that $\lambda_M(S, T) = \infty$ if $S \cap T \neq \emptyset$). The following version of Menger's theorem holds for mixed hypergraphs:

Theorem 1.3. *Let $M = (V, \mathcal{A})$ be a mixed hypergraph, and $S, T \subseteq V$. Then*

$$\lambda_M(S, T) = \min\{\varrho_M(X) : T \subseteq X \subseteq V - S\}.$$

Definition 1.4. *If $M = (V, \mathcal{A})$ is a mixed hypergraph, $r \in V$ is a designated root node, and k, l are nonnegative integers, then we say that M **is (k, l)-arc-connected from** r if $\lambda_M(r, v) \geq k$ and $\lambda_M(v, r) \geq l$ for any $v \in V$.*

1.3.2 Set functions

By set functions we will usually mean functions of the form $p : 2^V \to \mathbb{Z} \cup \{\infty, -\infty\}$. Let us give the most important definitions for set functions needed in the thesis.

Let p be a set function. We will say that a set $X \subseteq V$ is p**-positive** if $p(X) > 0$. We will widely use the notation $M_p = \max\{p(X) : X \subseteq V\}$. Any function $m : V \to \mathbb{R}$ also induces a set function (that will also be denoted by m) with the definition $m(X) = \sum_{v \in X} m(v)$ for any $X \subseteq V$. This notation together with the simplification $p(\{x\}) = p(x)$ introduced earlier for a set function p can again cause some confusion, but we hope that after this remark it will not (i.e. if p is a set function then $p(X)$ is not necessarily equal to $\sum_{v \in X} p(v)$).

A set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is called **skew-supermodular** if at least one of the following two inequalities holds for every $X, Y \subseteq V$:

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y), \tag{$\cap\cup$}$$
$$p(X) + p(Y) \leq p(X - Y) + p(Y - X). \tag{$-$}$$

A set function is **symmetric** if $p(X) = p(V - X)$ for every $X \subseteq V$. Symmetric skew-supermodular functions will be very important in this thesis. If $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is a set function and $(\cap\cup)$ holds for some sets $X, Y \subseteq V$ then we say that X **and** Y **satisfy** $(\cap\cup)$, or shortly that $X (\cap\cup) Y$: if we don't explicitly say which function is meant then we always mean p. The same notation is used for $(-)$. Note that $X(\cap\cup)Y$ is equivalent with $X(-)\overline{Y}$.

A set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is called **supermodular** if it satisfies $(\cap\cup)$ for any pair X and Y. A set function b is called **submodular** if $-b$ is supermodular. The most important example of a symmetric submodular function in this thesis is the degree function of a hypergraph. A set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is called **negamodular** if it satisfies $(-)$ for any pair X and Y. A set function b is called **posimodular** if $-b$ is negamodular. Sometimes the supermodular inequality $(\cap\cup)$ does not hold for any pair of sets X, Y, only for special pairs. If p satisfies $(\cap\cup)$ for any properly intersecting pair X, Y, then we say that it is **intersecting supermodular**. If p satisfies $(\cap\cup)$ only for crossing pairs X, Y, then we say that p is **crossing supermodular**. **Intersecting or crossing submodular (negamodular, posimodular)** functions are defined analogously.

Let us weaken further the notions introduced above. We say that a function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is **positively skew-supermodular** if at least one of $(\cap\cup)$ or $(-)$ holds for any $X, Y \subseteq V$ with $p(X), p(Y) > 0$. Note that we do not require that such a function is nonnegative, unlike it is usually assumed in the literature. For a set function p let $p^+(X) = \max(p(X), 0)$ for any $X \subseteq V$: if p is skew-supermodular then p^+ is positively skew-supermodular. We can also generalize the notion of crossing supermodular functions: a set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is called **positively crossing supermodular** if it satisfies $(\cap\cup)$ for any crossing pair X and Y with $p(X), p(Y) > 0$. However positively crossing (or skew-) supermodular functions cannot be handled algorithmically: this will be detailed later. Fortunately, in our applications we will only meet crossing supermodular and skew-supermodular functions. We can define **positively crossing negamodular** functions analogously.

A symmetric, (positively) crossing supermodular function p will satisfy both $(\cap\cup)$ and $(-)$ for an arbitrary (p-positive) crossing pair $X, Y \subseteq V$. An important observation is the following: if p is (positively) skew-supermodular and H is a hypergraph, then so is $p - d_H$. Similar statement holds if p is (positively) crossing supermodular or (positively) crossing negamodular etc.

Let $q : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a set function. Define **the complement of** q as $\overline{q}(X) = q(\overline{X})$ and **the symmetrized of** q by $q^s(X) = \max\{q(X), q(\overline{X})\}$ for any $X \subseteq V$: note that q^s is a symmetric function. Observe that a hypergraph H covers q if and only if H covers q^s.

Note that a crossing supermodular (or crossing negamodular) function is not necessarily skew-supermodular: we cannot ensure any of $(\cap\cup)$ and $(-)$ for co-disjoint sets (i.e. when $X \cup Y = V$). However the symmetrized of such functions is already skew-supermodular. The following claim is easy to prove, the proof is left to the reader.

Claim 1.5. *The symmetrized of a crossing supermodular, crossing negamodular or skew-supermodular function is (symmetric and) skew-supermodular.* □

Similar statement holds for the symmetrized of positively crossing supermodular, crossing negamodular or skew-supermodular functions: their symmetrized will be positively skew-supermodular. We will often use the following strengthening of $(\cap\cup)$ and $(-)$ implied by (1.1) and (1.2). Assume that p is of form $p = p_0 - d_G$ with some positively skew-supermodular function p_0 and graph $G = (V, E)$. For two p_0-positive subsets X and Y of V, at least one of the following two inequalities holds.

$$p(X) + p(Y) \le p(X \cap Y) + p(X \cup Y) - 2d_G(X, Y), \tag{1.5}$$
$$p(X) + p(Y) \le p(X - Y) + p(Y - X) - 2\overline{d}_G(X, Y). \tag{1.6}$$

Again, if p_0 is symmetric and positively crossing supermodular and X, Y are crossing and p_0-positive, then the reader can see that both (1.5) and (1.6) holds. The following similar statement is implied by (1.3) and (1.4).

Lemma 1.6. *Assume that p is of form $p = p_0 - d_H$ with some symmetric positively skew-supermodular function p_0 and hypergraph $H = (V, \mathcal{E})$. Let $X, Y \subseteq V$.*

(i) If X and Y satisfy $(\cap\cup)$ for p_0, but there is a hyperedge $e \in \mathcal{E}$ such that $e \cap (X - Y) \neq \emptyset, e \cap (Y - X) \neq \emptyset$, and e intersects at most one of $X \cap Y$ and $\overline{X \cup Y}$, then $p(X) + p(Y) < p(X \cap Y) + p(X \cup Y)$.

(ii) If X and Y satisfy $(-)$ for p_0, but there is a hyperedge $e \in \mathcal{E}$ such that $e \cap (X \cap Y) \neq \emptyset, e \cap (\overline{X \cup Y}) \neq \emptyset$, and e intersects at most one of $X - Y$ and $Y - X$, then $p(X) + p(Y) < p(X - Y) + p(Y - X)$. □

The following property of a positively crossing supermodular function will be very useful.

Claim 1.7. *If q is a positively crossing supermodular function, and X, Y are q-positive crossing sets with $q(Y) \ge q(X \cap Y)$ then $q(X) \le q(X \cup Y)$.*

The following claim generalizes the preceding one, it can be proved by a simple induction.

Claim 1.8. *Let q be a positively crossing supermodular function and $X_1, \dots, X_k$ be q-positive subsets of V so that X_j crosses $\bigcup_{i=1}^{j-1} X_i$ and $q(X_j \cap (\bigcup_{i=1}^{j-1} X_i)) \le q(X_j)$ for any $j = 2, \dots, k$. Then $q(\bigcup_1^{j-1} X_i) \le q(\bigcup_1^j X_i)$ for any $j = 2, \dots, k$. Consequently $q(X_1) \le q(\bigcup_1^j X_i) \le q(\bigcup_1^k X_i)$ for any $j = 1, 2, \dots, k$.*

Claim 1.8 will be usually applied under much simpler circumstances in the following form (in many cases i_j will just be 1 for every j when we apply Claim 1.9).

Claim 1.9. *Let $\{W_1, \ldots, W_k\}$ be a subpartition of V so that $\bigcup_1^k W_i \neq V$, and for every $j = 2, \ldots, k$ there exists an i_j such that $1 \leq i_j < j$, $q(W_{i_j}) = 1$, and $q(W_{i_j} \cup W_j) \geq 1$. Then $q(\bigcup_1^k W_i) \geq 1$.*

Proof. Apply Claim 1.8 for the sets $W_{i_j} \cup W_j$. □

On the other hand, if q is positively crossing negamodular then an important observation is the following.

$$\text{If } X, Y \subseteq V \text{ are crossing, } q(X) = \overline{q}(Y) = M_q > 0, \text{ then } q(X \cap Y) = M_q \text{ and } \overline{q}(X \cup Y) = M_q. \tag{1.7}$$

The following claim generalizes this statement in one direction. It is proved by a simple induction.

Claim 1.10. *Let $q : 2^V \to \mathbb{Z}$ be crossing negamodular and assume that $X_0, X_1, X_2, \ldots, X_t$ are subsets of V (where $t \geq 0$) such that $\overline{q}(X_0) = q(X_1) = q(X_2) = \cdots = q(X_t) = M_q > 0$ and X_i crosses $X_0 \cup \bigcup_{j<i} X_j$ for any $i = 1, 2, \ldots, t$. Then $\overline{q}(X_0 \cup \bigcup_{j \leq t} X_j) = M_q$.* □

1.4 Oracles

In our abstract algorithms we will usually deal with some set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$. However it is usually not allowed to enumerate all the function values for every $X \subseteq V$. So we will think of the function as something available **through an oracle**. But what kind of questions can the oracle answer for us? The most straightforward idea is a function evaluation oracle: we pass it a subset $X \subseteq V$ and it tells us the value $p(X)$. However this will usually not be sufficient for us: with this we cannot even decide whether the function takes finite values at all. There are other problems with this oracle, too, therefore we will need a more clever oracle. One can think of many types of oracles: we will introduce here two convenient ones. The first one is the simple function evaluation oracle, and the second is a tricky one, which will always be sufficient for our purposes. Then we will show the relationship between these two oracles for different types of functions. We say that a hypergraph $H = (V, \mathcal{E})$ is **given explicitly** if it is encoded by enumerating the (different) hyperedges as vectors in $\{0, 1\}^V$ and for every such hyperedge its multiplicity is given by a binary integer.

Definition 1.11. *An **evaluation oracle** for a set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ takes as input a subset X of V and it returns the function value $p(X)$.*

Definition 1.12. *A* ***maximizing oracle*** *for a set function* $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ *takes as input a hypergraph* H *(given explicitly) and a vector* $x \in \mathbb{R}_+^V$ *and it returns* $\max\{p(Z) - d_H(Z) - x(Z) : Z \subseteq V\}$ *and it also gives a set* $Z \subseteq V$ *that maximizes this quantity.*

This second oracle appears also in [5] and [29]. Our algorithms will usually use a maximizing oracle for p^s: note that if we have a maximizing oracle for p and one for $\overline{p}$, then we can implement a maximizing oracle for p^s using these two oracles. Let us give some motivation why the maximizing oracle is of the form given above. We will often want to decide whether a given $x \in \mathbb{Z}_+^V$ satisfies (1.8) or not, because it is a necessary condition on the existence of a hypergraph covering p with degree-sequence x: this can be done with a maximizing oracle for p^s by calling it with x and the empty hypergraph. Similarly, in many cases we have already chosen some hyperedges that we want to include in the hypergraph covering some function p_0: if we denote this hypergraph by H then a degree-specification x for the remaining hyperedges has to satisfy (1.8) with $p = p_0 - d_H$ and this again can be decided with a maximizing oracle for p_0^s.

$$x(Z) \geq p^s(Z) \text{ for every } Z \subseteq V. \tag{1.8}$$

Note that usually we cannot test the properties of the function p claimed by the oracle (e.g. supermodularity, skew-supermodularity, symmetry etc.), since that would need too many oracle calls. Therefore in this model our algorithms will only give correct answers if they can access the appropriate oracle and the function has the claimed properties.

Note that the maximizing oracle can clearly provide us $M_p = \max\{p(X) : X \subseteq V\}$. However it is not clear whether we can implement the evaluation oracle with help of a maximizing oracle: if we want to determine $p(X)$ for some $X \subseteq V$ then we need to find an $x \in \mathbb{R}_+^V$ and a hypergraph H such that X becomes the only maximizer of $\max\{p(Z) - d_H(Z) - x(Z) : Z \subseteq V\}$. However if $p(X) = -\infty$ then this never happens.

Claim 1.13. *If* $p(X) \geq 0$ *then we can evaluate* $p(X)$ *with a maximizing oracle for* p*. If the set function* p *has only finite values and we are given some lower bound* $L \leq \min\{p(Z) : Z \subseteq V\}$*, then an evaluation oracle can be implemented with help of a maximizing oracle.*

Proof. Let $M = M_p + 1$ for the first statement and $M = M_p + 1 - L$ for the second. Let H be the disjoint union of an arbitrary tree with node set X and one with node set $V - X$ and let the multiplicity of every edge of H be M. Let furthermore $x(v) = M$ for every $v \in V - X$ and $x(v) = 0$ for every $v \in X$. Call the maximizing oracle to maximize $p - d_H - x$. One can see that $p(Z) - d_H(Z) - x(Z) \leq p(Z) - M \leq M_p - M$, if $Z \subseteq V$ is different from X, which is in both statements strictly less than $p(X)$. Thus the unique maximizer is X and the maximum is $p(X)$. $\square$

It is more important to characterize the function classes in which we can (or we cannot) implement the *maximizing oracle* with polynomially many calls to the *evaluation oracle*.

First we show that for positively skew-supermodular functions the maximizing oracle cannot be reduced to the evaluation oracle. This is true even for symmetric positively crossing supermodular functions. Consider the example $p(X) = 1$ if $X = X_0$ or $X = V - X_0$ for some fixed X_0 (and 0 otherwise): this function is clearly symmetric and positively crossing supermodular, but we need exponentially many calls to the evaluation oracle to decide whether $M_p > 0$ or not.

We have similar problems if the function can take $-\infty$ as value, too. The same example (but with $-\infty$ instead of zero values) shows that the maximizing oracle cannot be implemented with polynomially many calls to the evaluation oracle even if p is symmetric and crossing supermodular (but can take the value $-\infty$, too). We note that maximizing a (crossing) supermodular function having $-\infty$ among the function values (and thus implementing the maximizing oracle for such functions) can be solved if we have a little stronger oracle than the function evaluation oracle given above: this can be found in [22].

On the other hand, for a finitely valued crossing supermodular function p, the maximizing oracle can be implemented through general **submodular function minimization techniques** (see e.g. [27] or [40]). This is true since the function $p - d_H - x$ is still crossing supermodular (if H is an arbitrary hypergraph and $x \in \mathbb{R}^V_+$) and the maximization of a crossing supermodular function can be reduced to the maximization of a (fully) supermodular function.

It is a challenging open problem to decide whether one can implement the maximizing oracle for a **finitely valued skew-supermodular function**. This is essentially the same problem as the problem of maximizing such a function. The simplest open question is whether we can maximize a **finitely valued intersecting negamodular function** (with polynomially many calls to the function evaluation oracle).

In Section 2.1 we will show how to implement the maximizing oracle for the edge-connectivity requirement functions appearing in our applications.

1.5 G-polymatroids and their intersections

In this section we give some background on (integer) g-polymatroids, of course without aiming for completeness. For a complete introduction we refer the reader to [22]. In the rest of Section 1.5 we assume that the set functions $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ and $b : 2^V \to \mathbb{Z} \cup \{+\infty\}$ satisfy that $p(\emptyset) = b(\emptyset) = 0$.

We say that the functions p, b form a **strong pair**, if p is supermodular, b is submodular and they satisfy the following **cross-inequality** for any pair $X, Y \subseteq V$:

$$b(X) - p(Y) \geq b(X - Y) - p(Y - X). \tag{1.9}$$

If p, b form a strong pair then the following polyhedron is called a **generalized polymatroid**, or shortly a **g-polymatroid**:

$$Q(p, b) = \{x \in \mathbb{R}^V : \; p(Z) \leq x(Z) \leq b(Z) \text{ for any } Z \subseteq V\}. \tag{1.10}$$

It is convenient to consider the empty set as a g-polymatroid, too, though it cannot be defined with a strong pair. The notion of g-polymatroids was introduced by Frank in [18] as a convenient common generalization of polymatroids, contrapolymatroids, base polyhedra and submodular polyhedra. Note that the system (1.10) has an exponential size in $|V|$. Therefore we will think of it as a system that is given only implicitly: we can access the functions p and b through some oracle. A basic question is whether we can test membership in $Q(p, b)$ and whether we can optimize a linear objective function over $Q(p, b)$ in polynomial time. We note that by the ellipsoid method due to Grötschel, Lovász and Schrijver [25], these two problems (optimization and separation) are equivalent, therefore we will usually just show that we can solve one of them. Frank proved several nice properties of g-polymatroids. We will need the following.

Theorem 1.14. ***(A. Frank, [18])*** *A g-polymatroid is an integer polyhedron. If the functions p and b are given with an evaluation oracle and (p, b) is a strong pair then one can optimize a linear objective function over $Q(p, b)$ in polynomial time.*

It is true that the formula (1.10) defines a g-polymatroid under more general circumstances, too. We will need the following result. The functions p, b form a **weak pair** if $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is intersecting supermodular, $b : 2^V \to \mathbb{Z} \cup \{+\infty\}$ is intersecting submodular and they satisfy the cross-inequality for properly intersecting set pairs.

Theorem 1.15. ***(A. Frank, [18])*** *The polyhedron defined by (1.10) is a (possibly empty) g-polymatroid even is p and b form a weak pair.*

Since we will define g-polymatroids with even weaker set functions we will need the stronger maximizing oracle in order to be able to find elements of these g-polymatroids.

The following proposition is a corollary of Theorem 1.15.

Theorem 1.16. *For an integer g-polymatroid $Q \subseteq \mathbb{R}^V$, functions $l : V \to \mathbb{Z} \cup \{-\infty\}$ and $u : V \to \mathbb{Z} \cup \{\infty\}$, and numbers $\alpha \in \mathbb{Z} \cup \{-\infty\}, \beta \in \mathbb{Z} \cup \{\infty\}$, the polyhedron*

$$Q \cap \{x \in \mathbb{R}^V : l(v) \leq x(v) \leq u(v) \textit{ for every } v \in V, \;\; \alpha \leq x(V) \leq \beta\}$$

is a (possibly empty) integer g-polymatroid.

An important special case of a g-polymatroid in this thesis is the **contrapolymatroid**, which is the following polyhedron for a monotone supermodular function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$:

$$C(p) = \{x \in \mathbb{R}^V : x(Z) \geq p(Z)\ \forall Z \subseteq V, x \geq 0\}. \tag{1.11}$$

Note that the monotonicity of p together with $p(\emptyset) = 0$ implies that $p \geq 0$. We point out that different monotone supermodular functions define different contrapolymatroids, since the polyhedron $C(p)$ determines its defining monotone supermodular function p by the following relation:

$$p(Z) = \min\{x(Z) :\ x \in C(p)\}. \tag{1.12}$$

Again, $C(p)$ defined by (1.11) is a (nonempty) contrapolymatroid under more general circumstances, too, by the following result.

Theorem 1.17. ***(A. Frank, [19], [2])*** *If $p : 2^V \to \mathbb{Z}_+$ is positively skew-supermodular then $C(p) = C(p')$ with the (uniquely defined) monotone supermodular function*

$$p'(X) = \max\{\sum_{i=1}^{t} p(X_i) :\ X_1, X_2, \ldots, X_t \textit{ is a subpartition of } X\}. \tag{1.13}$$

The intersection of two g-polymatroids is again an integer polyhedron, in fact it is a special **submodular flow polyhedron**. We omit the definition of submodular flow polyhedra since we will not need them in this thesis, we will only use the following result.

Theorem 1.18. ***(A. Frank, [18])*** *The intersection of two g-polymatroids is an integer polyhedron.*

Chapter 2

Edge-connectivity augmentation by adding (hyper)edges

The most natural approach to edge-connectivity augmentation is the following: we are given a graph or hypergraph and we want to **add edges or hyperedges** to it in order **to meet some edge-connectivity requirements**. One possible objective function is to minimize the total size of the hyperedges needed. The main concern of this thesis is indeed this notion of edge-connectivity augmentation (note however that we will use a different notion of edge-connectivity augmentation in Chapter 6). Sometimes we will augment directed structures (i.e. mixed graphs or mixed hypergraphs), though we emphasize that **we will always want to add undirected edges or hyperedges** to our initial (possibly directed) graph or hypergraph. This problem can be formulated as the problem of **covering some special set function with a graph or hypergraph** as we will see shortly. In this thesis the most general class of set functions that we consider will be *positively skew-supermodular functions*. Because of its importance let us state this covering problem explicitly.

Problem 2.1 (Covering a symmetric, positively skew-supermodular function with (hyper)edges). *Given a symmetric, positively skew-supermodular function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ with a maximizing oracle, find a (hyper)graph H of minimum total size covering p.*

The (hyper)graph H to be found will also be called the *augmenting (hyper)graph*. By the remarks on the symmetrized of a set function, symmetry matters only in the following sense: if p is not symmetric then we need to have access to a maximizing oracle for p^s. Note that the problem has already two versions, depending on whether we allow arbitrary hyperedges or only graph edges in covering our function. The function p will also be called the **requirement function**, or the **deficiency function**.

In the subsequent sections first we give the examples that motivate Problem 2.1 by showing that it is indeed a **general framework for edge-connectivity augmentation**. In the following examples the objective is always to minimize the total size of the augmenting (hyper)graph H (if H has to be a graph then this is the same as minimizing the number of edges in H). We will discuss different objectives in the Section 2.2.

2.1 Examples: edge-connectivity requirement functions

In the following sections we will show applications where the augmentation problem can be formulated as covering a skew-supermodular set function. There are always different versions of the problems depending on further constraints: we lay the emphasis on the deficiency function here only. First we give a construction that will be used in the proofs below.

Definition 2.2. *Given a mixed hypergraph $M = (V, \mathcal{A})$, an undirected hypergraph $H = (V, \mathcal{E})$ and a function $x : V \to \mathbb{R}_+$, we define the mixed hypergraph $M' = M'(M, H, x)$ as follows. Introduce a new node $z \notin V$ and let the vertex set of M' be $V + z$. Connect z with every $v \in V$ with a graph edge having capacity $x(v)$: the arc set of M' consists of the (disjoint) union of $\mathcal{E}$ and $\mathcal{A}$ and the (capacitated) graph edges incident to z.*

Note that if M itself is an undirected hypergraph in the above definition, too, then so is M'.

2.1.1 Global edge-connectivity augmentation

The most natural question in edge-connectivity augmentation is the following.

Problem 2.3 (Global edge-connectivity augmentation problem). *Given a graph or hypergraph $H_0 = (V, \mathcal{E}_0)$ and a positive integer k, find a (hyper)graph $H = (V, \mathcal{E})$ such that $H_0 + H$ is k-edge-connected.*

Note that the problem has many versions depending on whether H_0 and H is a graph or a hypergraph. By Menger's theorem (Theorem 1.2) this is equivalent to the problem of covering the following set function:

$$p(X) := \begin{cases} k - d_{H_0}(X) & \text{for any nonempty } X \subsetneq V, \\ 0 & \text{for } X = \emptyset \text{ and } X = V. \end{cases} \tag{2.1}$$

This function p has very nice properties: it is *symmetric and crossing supermodular*. Thus the **global edge-connectivity augmentation problem is indeed a special case of Problem 2.1 with the symmetric crossing supermodular function** defined in (2.1).

This problem has already many versions. The first version is when H_0 is a graph and H has to be a graph, too: this is the **global edge-connectivity augmentation problem of graphs**. This problem was solved by Watanabe and Nakamura in [45]. The second version is when H_0 is a hypergraph, but we want to augment it only with graph edges: this problem is the **global edge-connectivity augmentation problem of hypergraphs with graph edges**. This problem was solved by Bang-Jensen and Jackson in [4]. Our contribution to these topics is simplified proofs that enable us to handle generalizations, too. See Sections 4.2.2 and Chapter 5 for more details. The version when H may contain hyperedges can be solved under more general circumstances: see Section 2.1.4.

2.1.2 Global arc-connectivity augmentation of mixed hypergraphs

A bit more difficult problem arises when we want to augment a mixed (hyper)graph, instead of an undirected one (however we only want to add *undirected (hyper)edges* to it!).

Problem 2.4 (Global arc-connectivity augmentation of mixed hypergraphs). *Let $M = (V, \mathcal{A})$ be a mixed hypergraph, $r \in V$ be a designated root node, and k, l be nonnegative integers. Find a (hyper)graph $H = (V, \mathcal{E})$ of minimum total size such that $\lambda_{M+H}(r, v) \geq k$ and $\lambda_{M+H}(v, r) \geq l$ for any $v \in V$ (i.e. $M + H$ is (k, l)-arc-connected from r).*

Again, the problem has many versions. Let us define the set function $q = q_{M,r,k,l}$ by

$$q_{M,r,k,l}(X) := \begin{cases} k - \varrho_M(X) & \text{for any } X \neq \emptyset \text{ with } r \notin X, \\ l - \varrho_M(X) & \text{for any } X \neq V \text{ with } r \in X \\ 0 & \text{for } X = \emptyset \text{ and } X = V. \end{cases} \tag{2.2}$$

The set function q is crossing supermodular, but it is not symmetric. For an undirected hypergraph H one can check using the directed version of Menger's theorem (Theorem 1.3) that $M + H$ is (k, l)-arc-connected from r if and only if d_H covers q or equivalently $p = q^s$, which is a special symmetric skew-supermodular set function. Thus the **global arc-connectivity augmentation of mixed hypergraphs is indeed a special case of Problem 2.1 where p is the symmetrized of a crossing supermodular function** q defined in (2.2).

Lemma 2.5. *The maximizing oracle can be implemented for the function $p = q^s_{M,r,k,l}$ introduced in this section.*

Proof. Assume that we want to maximize $p - d_H - x$ for some hypergraph H and $x \in \mathbb{R}^V_+$. Construct the mixed hypergraph $M' = M'(M, H, x)$ as shown in Definition 2.2 and observe that $\lambda_{M'}(r,v)$ and $\lambda_{M'}(v,r)$ can be determined for any $v \in V$ with standard network flow techniques. The maximum of $p - d_H - x$ is equal to $\max_{v\in V}\{k - \lambda_{M'}(r,v), l - \lambda_{M'}(v,r)\}$ and the maximizer can be found similarly. □

If M is a mixed graph and we want to add graph edges to it then this problem was solved by Bang-Jensen, Frank and Jackson [2]: for a simplified proof see Section 4.2.2. If M is a mixed hypergraph and we only allow the addition of graph edges then the problem is unsolved. If we also allow hyperedges then the problem can be solved even with more restrictions, for example we can require that the augmenting hypergraph has to be nearly uniform. See details of this version in Section 3.4.3.

2.1.3 Node-to-area edge-connectivity augmentation

A different set function arises when we want to solve a **node-to-area connectivity augmentation problem**. The problem is the following.

Problem 2.6 (Node-to-area connectivity augmentation problem). *Given a (hyper)graph $H_0 = (V, \mathcal{E}_0)$, a collection of subsets $\mathcal{W}$ of V and a function $r : \mathcal{W} \to \mathbb{Z}_+$, our aim is to find a (hyper)graph H of minimum total size such that*

$$\lambda_{H_0+H}(x, W) \geq r(W) \text{ for any } W \in \mathcal{W} \text{ and } x \in V. \tag{2.3}$$

Some versions of this problem will be detailed below. Note that the global edge-connectivity augmentation problem is a special case where the areas are all the singletons and the requirement of every area is the same.

Let us show why this problem is a special case of Problem 2.1. Define

$$R_{N2A}(X) = \max\{r(W) : W \in \mathcal{W},\ W \cap X = \emptyset\} \text{ for any } \emptyset \neq X \subseteq V \text{ and } R_{N2A}(\emptyset) = 0. \tag{2.4}$$

This is a monotone decreasing function: by that we mean that $R_{N2A}(X) \geq R_{N2A}(Y)$ for any $\emptyset \neq X \subseteq Y$. This easily implies that R_{N2A} is crossing negamodular (in fact it is even intersecting negamodular), and by Menger's theorem (Theorem 1.2) a (hyper)graph H is a feasible solution to the node-to-area connectivity augmentation problem if and only if d_H covers $q = R_{N2A} - d_{H_0}$ or equivalently $p = q^s$. Thus the **node-to-area connectivity augmentation problem is indeed a special case of Problem 2.1 where p is the symmetrized of a crossing negamodular function q.** In fact the crossing negamodular function q is of special form, since it is the difference of a monotone decreasing function and a symmetric submodular function.

The **node-to-area connectivity augmentation problem in graphs** is the special case of this problem when H_0 is a graph and H has to be a graph, too. This problem was introduced by Ishii and Hagiwara in [26]. It is in general NP-complete (even if H_0 is the empty graph and $r(W) = 1$ for every $W \in \mathcal{W}$): because of the beauty and simplicity of the proof given by Zoltán Király we present the proof of this.

Theorem 2.7. *The (general) node-to-area connectivity augmentation problem in graphs is NP-complete, even if the graph to be augmented is the empty graph and the requirement is 1 for every area.*

Proof (Z. Király). We reduce the *three dimensional matching* problem (Problem SP1 in [23]) to our problem. Let $H = (V, \mathcal{E})$ be an instance of the three dimensional matching problem (that is, H is a 3-uniform hypergraph, where 3 divides $|V|$ and the question is whether V can be covered with disjoint hyperedges of H). Let $\mathcal{W} = \{V - X \subseteq V : |X| \in \{1, 2, 3\}$ and $X \notin \mathcal{E}\}$ and let $r : \mathcal{W} \to \mathbb{Z}_+$ be defined by $r(W) = 1$ for each $W \in \mathcal{W}$. One can check that the optimal solution of the node-to-area connectivity augmentation problem in graphs defined by $\mathcal{W}, r$ and the empty hypergraph H_0 contains exactly $2|V|/3$ edges (its components are paths of length 2) if and only if H contains a three dimensional matching. $\square$

Since the node-to-area connectivity augmentation problem in graphs is NP-complete, the authors of [26] assume that $r \geq 2$ and surprisingly the problem becomes tractable: they give a polynomial time algorithm that solves it. Their proof was simplified by Szigeti and Grappe in [24]. We further simplify a part of this proof: we show that a greedy algorithm fails only slightly for this problem in Section 4.3.2. In the thesis the *node-to-area connectivity augmentation problem in graphs* will mean the case when $r \geq 2$ to distinguish from the *general node-to-area connectivity augmentation problem in graphs*, which is NP-complete.

The **node-to-area connectivity augmentation problem in hypergraphs** is the case when H_0 and H can be arbitrary hypergraphs. We will discuss this problem in Section 3.4.2. This version can be solved in polynomial time even without the assumption that $r \geq 2$.

We will consider a third version of this problem: assume that H_0 is a hypergraph of rank at most γ and we can only add hyperedges of size at most γ to it: this is the **rank-respecting node-to-area connectivity augmentation problem in hypergraphs**. As the $\gamma = 2$ case shows, this is again NP-complete in general, so we need the $r(W) \neq 1$ requirement, but under this assumption we solve the problem in Section 4.4.3.

A similar proof to that of Lemma 2.5 justifies the following lemma.

Lemma 2.8. *The maximizing oracle can be implemented for the function $p = (R_{N2A} - d_{H_0})^s$ introduced in this section.* $\square$

Proof. Assume that we want to maximize $p - d_H - x$ for some hypergraph H and $x \in \mathbb{R}^V_+$. Construct the hypergraph $M' = M'(H_0, H, x)$ as shown in Definition 2.2. The maximum of $p - d_H - x$ is equal to $\max\{r(W) - \lambda_{M'}(W, v) : W \in \mathcal{W}, v \in V - W\}$ which (together with a maximizer) can be determined with standard network flow techniques. □

2.1.4 Local edge-connectivity augmentation

If we consider the local edge-connectivity of graphs or hypergraphs then we obtain a symmetric skew-supermodular deficiency function that is not the symmetrized of a crossing supermodular or crossing negamodular function. Let us introduce the problem.

Problem 2.9 (**Local edge-connectivity augmentation problem**). *Let H_0 be a (hyper)graph and let $r : V \times V \to \mathbb{Z}_+$ be a symmetric edge-connectivity requirement (i.e. $r(u, v) = r(v, u)$ for every $u, v \in V$). Our aim is to find a (hyper)graph H of minimum total size such that $\lambda_{H_0+H}(u, v) \geq r(u, v)$ for every pair of nodes u, v.*

This class of problems is called local edge-connectivity augmentation problems, since the requirement is locally defined for the pairs of nodes. Note that the global edge-connectivity augmentation is the special case when $r(u, v)$ is the same for any pair $u, v \in V$. The **local edge-connectivity augmentation of graphs** is the problem when H_0 is a graph and H has to be a graph, too. The **local edge-connectivity augmentation of hypergraphs** is the case when H_0 is a hypergraph and H can be an arbitrary hypergraph, too. We will also consider other versions to be defined later.

Let us define the set function R_{loc} as $R_{loc}(\emptyset) = R_{loc}(V) = 0$ and

$$R_{loc}(X) = \max\{r(u, v) : u \in X, v \notin X\} \text{ for any nonempty } X \subsetneq V. \tag{2.5}$$

By Menger's theorem $\lambda_{H_0+H}(u, v) \geq r(u, v)$ for every pair of nodes u, v if and only if $d_{H_0+H}(X) \geq R_{loc}(X)$ for any $X \subseteq V$, in other words if and only if H covers $p = R_{loc} - d_{H_0}$. An important observation is the following.

Theorem 2.10 (A. Frank [19]). *The function R_{loc} defined by (2.5) is skew-supermodular.*

Since the function R_{loc}, and therefore $R_{loc} - d_{H_0}$ is symmetric and skew-supermodular, the **local edge-connectivity augmentation problem is indeed a special case of Problem 2.1 where p is the symmetric skew-supermodular function of form $R_{loc} - d_{H_0}$ and R_{loc} is defined in** (2.5). We mention that the possibility that $r(u, v) = 1$ for certain pairs u, v again causes difficulties in some cases, so we will often assume that $r(u, v) \neq 1$ for any pair u, v, but here this is only a technical assumption and the complexity of the problem does not depend on it. The following lemma can again be proved similarly to Lemma 2.5.

Lemma 2.11. *The maximizing oracle can be implemented for the function $p = R_{loc} - d_{H_0}$ introduced in this section.*

Proof. Assume that we want to maximize $p - d_H - x$ for some hypergraph H and $x \in \mathbb{R}_+^V$. Construct the hypergraph $M' = M'(H_0, H, x)$ as shown in Definition 2.2. The maximum of $p(Z) - d_H(Z) - x(Z)$ is equal to $\max\{r(u, v) - \lambda_{M'}(u, v) : \ u, v \in V\}$ which (together with a maximizer) can be determined with standard network flow techniques. □

The local edge-connectivity augmentation of graphs was solved by Frank [19] using the splitting lemma of Mader [34]. Our contribution is a simple proof of this result. If H_0 is a hypergraph and we want to augment it with graph edges to meet local edge-connectivity requirements, then we obtain an NP-complete problem, as was shown in [16]. The case when H may contain arbitrarily large hyperedges was solved by Szigeti [42]. In Section 3.4.1 we generalize his results and show, for example, that one can find an optimal solution that is nearly uniform. If H can contain hyperedges but its rank cannot be bigger than that of H_0 then the problem was solved by Ben Cosh [15]: we present a simpler proof of this result in Section 4.4.1.

2.2 Polyhedra and various objective functions of the augmentation

As we have seen, the **edge-connectivity augmentation problems** that we consider here can be formulated as the problem of finding a (hyper)graph covering a positively skew-supermodular set function, i.e. a **covering problem**. Let us look at this problem in terms of the objective function.

The most simple objective function of edge-connectivity augmentation would be to minimize the number of edges of the augmenting graph (or the total size of the augmenting hypergraph): let us call this version of the problem the **minimum augmentation problem**. However it is more comfortable to speak about another problem, the so-called **degree specified augmentation problem**. This is the following: given a symmetric positively skew-supermodular set function p and a **degree specification** $m : V \to \mathbb{Z}_+$, the question is whether a graph (or hypergraph) G covering p exists that **satisfies the degree-specification** meaning that $d_G^+(v) = m(v)$ has to hold for every $v \in V$ (so this is not an optimization problem, but a decision problem). A **natural necessary condition** of the existence of the graph G is that $m(X) = \sum_{v \in X} m(v) \geq p(X)$ for any $X \subseteq V$, since $d_G(X) \leq \sum_{v \in X} d_G(v) \leq \sum_{v \in X} d_G^+(v) = \sum_{v \in X} m(v)$ for any $X \subseteq V$. This necessary condition and the nice properties of the function p allow us to reduce the minimum version (and even more general versions) of the augmentation problem to the degree-specified version. We say that the degree-specification $m \in \mathbb{Z}_+^V$ is **admissible** if $m(X) \geq p(X)$ for any $X \subseteq V$. If m is not admissible then a set X with $m(X) < p(X)$ will be called **deficient**.

For a set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ we have introduced the polyhedron

$$C(p) = \{x \in \mathbb{R}^V : x(Z) \geq p(Z)\ \forall Z \subseteq V, x \geq 0\}. \tag{2.6}$$

As it was mentioned in Section 1.5, this is an (integer) contrapolymatroid for a positively skew-supermodular function p. We note that, by the properties of a contrapolymatroid, a polynomial algorithm to the degree specified covering problem will give rise to a solution to the minimum version of the problem, and to more general versions, such as the minimum node-cost problem. By the properties of g-polymatroids (see Section 1.5), $\min\{x(V) : x \in C(p)\} = \min\{1 \cdot x : x \in C(p)\} = \max\{\sum_{X \in \mathcal{X}} p(X) : \mathcal{X}$ is a subpartition of $V\}$. Define the **Subpartition Lower Bound** by $SLB(p) = \max\{\sum_{X \in \mathcal{X}} p(X) : \mathcal{X}$ is a subpartition of $V\} = \min\{1 \cdot x : x \in C(p)\}$: this is obviously a lower bound on the minimum total size of any hypergraph covering p. We say that $m \in C(p) \cap \mathbb{Z}^V$ is **minimal** if $m' \in C(p) \cap \mathbb{Z}^V$, $m' \leq m$ implies that $m' = m$. By Edmonds' greedy algorithm and Theorem 1.17 we have the following.

Corollary 2.12. *For a vector $m \in C(p)$, m is minimal if and only if $m(V) = SLB(p)$.*

By the properties of a contrapolymatroid we can handle the following, **minimum node-cost covering problem**, too. This is the following: assume that we are given a cost function $c : V \to \mathbb{R}_+$ and the task is to find a (hyper)graph H covering p that minimizes $\sum_{v \in V} c(v) d_H(v)$. The minimum version of the covering problem corresponds to the case when $c(v) = 1$ for every $v \in V$. Note that we can assume about any optimal solution H of such a minimum node-cost covering problem that it does not contain singleton hyperedges, so $d_H(v) = d_H^+(v)$ for any $v \in V$.

The reader can ask why we consider exactly these objective functions. The answer is simple: these are the tractable ones. For example it is easy to see that the following simplest *minimum cost edge-connectivity augmentation problem in graphs* is already NP-complete: given a graph G_0 and a requirement $k \in \mathbb{Z}_+$, find a set of edges F of minimum cost such that $G + F$ is k-edge-connected, where the cost of choosing the edge uv is given for any $u, v \in V$. If $k = 2$ and G_0 is the empty graph then this problem is equivalent to the *Symmetric Travelling Salesman Problem*, as one can check. Let us define formally the considered versions of the problems in terms of the objective function.

Definition 2.13. *Let us be given the problem of covering a set function $p : 2^V \to \mathbb{R} \cup \{-\infty\}$ with a hypergraph H (where we might have further constraints on the hypergraph H). The*

minimum version *of the problem is to find a hypergraph H covering p of minimum total size. The* ***minimum node-cost version*** *of the problem is to find a (hyper)graph H covering p that minimizes $\sum_{v\in V} c(v)d_H(v)$ for a given cost function $c : V \to \mathbb{R}_+$. The* ***degree-specified version*** *is to find a hypergraph H covering p that also satisfies a given degree-specification $m \in \mathbb{Z}_+^V$.*

For sake of clarity let us outline Edmonds' greedy algorithm to find a vector $m \in C(p)$ that minimizes $\{\sum_{v\in V} c(v)x(v) : \ x \in C(p)\}$ for a positively skew-supermodular function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ (given by a maximizing oracle) and an arbitrary $c : V \to \mathbb{R}_+$ (i.e. the greedy algorithm for a contrapolymatroid). Determine $M_p = \max\{p(X) : \ X \subseteq V\}$ and set initially $m(v) = M_p$ for every $v \in V$. Order the elements of V such that $c(v_1) \geq c(v_2) \geq \cdots \geq c(v_n)$. Starting with $i = 1$ successively decrease $m(v_i)$ by a maximum possible value that maintains $m \in C(p)$: note that this maximum value can be determined by a logarithmic search. By the well known results of Edmonds [17], the resulting m will be an integer minimizer of $\{\sum_{v\in V} c(v)x(v) : \ x \in C(p)\}$. This algorithm can clearly be implemented to run in polynomial time if a maximizing oracle is available for p and it shows the ideas of the reduction of the *minimum node-cost covering problem* to the *degree-specified covering problem.*

Notice that using Edmonds' greedy algorithm we can implement the evaluation oracle for the monotone supermodular function p' defined in (1.13) that defines the same contrapolymatroid $C(p)$, if the positively skew-supermodular function p is given with a maximizing oracle. This is true since $p'(Z) = \min\{x(Z) : \ x \in C(p)\}$ by (1.12), therefore to evaluate $p'(Z)$ we only need to minimize $\chi_Z x$ over $C(p)$.

Because of the arguments given above we will always speak about the degree-specified covering problem and we will usually only state the solution of the minimum version as consequences.

2.3 Covering with graph edges: the splitting-off technique

When we want to cover a set function by a **degree specified graph** then the usual technique is **splitting-off**. The splitting-off operation was originally introduced by Lovász [33] and subsequently developed further by Mader [34] and others (we mention that a directed version of splitting-off can be defined, too, but we will only use undirected splitting-off). In the rest of Section 2.3 let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric, positively skew-supermodular function that satisfies $p(\emptyset) \leq 0$ and $m : V \to \mathbb{Z}_+$ a nonnegative function satisfying $m(X) \geq p(X)$ for any $X \subseteq V$ (i.e. an integer element of $C(p)$, or with other words an admissible degree-specification). We would like to decide whether there is a graph G covering p that satisfies $d_G^+(v) = m(v)$ for every $v \in V$.

For a node $v \in V$ we say that v is **positive** if $m(v) > 0$, and **neutral** otherwise. The set of positive nodes will be denoted by V^+. Assume $u, v \in V^+$ are two positive nodes (possibly $u = v$, but then $m(u) \geq 2$ is assumed). The operation **splitting-off** (or shortly **splitting**) **at** $\boldsymbol{u}$ **and** $\boldsymbol{v}$ consists of replacing m by m' and p by p' where

$$m' = m - \chi_{\{u\}} - \chi_{\{v\}} \text{ and } p'(X) = \begin{cases} p(X) - 1 & \text{if } |X \cap \{u, v\}| = 1, \\ p(X) & \text{otherwise.} \end{cases} \tag{2.7}$$

(Note that $p' = p - d_{(V,\{uv\})}$ where $d_{(V,\{uv\})}$ denotes the cut function of the graph $(V, \{uv\})$ having only one edge between u and v, therefore p' is positively skew-supermodular, too.) The edge uv will be called a **split edge**. One can observe that this is indeed the usual notion of splitting-off: if we introduce a graph $K = (V+s, E)$ with every edge of E incident to s and $d_K(s, v) = m(v)$ for any $v \in V$ then we are back at the well known splitting-off operation.

If $m'(X) \geq p'(X)$ for any $X \subseteq V$ then we say that **the splitting off is admissible** (or **the pair** $\boldsymbol{u, v}$ **is admissible**). A set X is **dangerous** if $m(X) - p(X) \leq 1$ and it is called **tight** if $m(X) - p(X) = 0$. The following claim can be easily verified.

Claim 2.14. *If $m \in C(p) \cap \mathbb{Z}^V$ and $u, v \in V^+$ then the splitting-off at u and v is admissible if and only if there is no dangerous set X containing both u and v.* □

We will also say that such a dangerous set $\boldsymbol{X}$ **blocks the splitting at** $\boldsymbol{u}$ **and** $\boldsymbol{v}$, or simply that $\boldsymbol{X}$ **blocks** $\boldsymbol{u}$ **and** $\boldsymbol{v}$.

We will sometimes need to do the **inverse of a splitting-off**. So assume that e is a split edge. The **unsplitting** operation of e is simply the reverse of the splitting-off operation: $m^e = m + \chi_{\{u\}} + \chi_{\{v\}}$ and $p^e = p + d_{(V,\{(uv)\})}$. Of course, this operation is always admissible, that is $m^e \in C(p^e)$, if m was admissible (since $m(X) - p(X) \leq m^e(X) - p^e(X)$ for any set X).

An **admissible splitting-off sequence** is a sequence of splitting-offs, where each step is admissible (where note that the functions p and m are always updated after each step). It is important to observe that the order of such a sequence can be arbitrarily changed: if the admissible splitting-off at u, v is followed by the admissible splitting-off at x, y, then first performing the splitting-off at x, y and then at u, v is also an admissible splitting-off sequence (this is because the unsplitting is always admissible). A **complete admissible splitting-off** is an admissible splitting-off sequence which decreases $m(V)$ to zero. Note that there exists a graph covering the function p and satisfying the degree-specification m if and only if there exists a complete admissible splitting-off.

Now we give some lemmas on splitting-off that will be needed later.

Claim 2.15. *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric function and $m \in C(p)$. If $T \subseteq V$ is tight and $D \subseteq V$ is dangerous satisfying $T \cup D = V$ then $m(T \cap D) = 0$.*

Proof. We can assume that $m(V) > 0$ and $T \cap D \neq \emptyset$, otherwise the claim is trivially true. Assuming that $m(T \cap D) > 0$ and using that $p(D) = p(T - D)$ and $p(T) = p(D - T)$ by the symmetry of p, we get a contradiction from the following inequalities: $p(T) + p(D) = p(T-D)+p(D-T) \leq m(T-D)+m(D-T) \leq m(T)-1+m(D)-1 \leq p(T)-1+p(D)$. $\square$

Lemma 2.16. *Assume that $p : 2^V \to \mathbb{Z}\cup\{-\infty\}$ is a symmetric, positively skew-supermodular function and $m \in C(p)$ is minimal. If the pair $u, v \in V^+$ is admissible, then splitting-off this pair the subpartition lower bound decreases by two, that is $SLB(p') = SLB(p) - 2$ for the modified function $p' = p - d_{(V,\{uv\})}$.*

Proof. Since $m' = m - \chi_{\{u\}} - \chi_{\{v\}} \in C(p')$, certainly $SLB(p') \leq SLB(p) - 2 = m(V) - 2 = m'(V)$. On the other hand, if $SLB(p) = \sum_{X\in\mathcal{X}} p(X)$ for some subpartition $\mathcal{X}$ of V, then u and v must be in two different members of $\mathcal{X}$ by the minimality of m and the admissibility of u, v, therefore $SLB(p') \geq \sum_{X\in\mathcal{X}} p'(X) = SLB(p) - 2$. $\square$

We mention that the minimality is indeed necessary in Lemma 2.16. As an example, consider the local edge-connectivity augmentation of a $C_5 = (\{x_1, x_2, x_3, x_4, x_5\}, \{x_1x_2,\ x_2x_3,\ x_3x_4,\ x_4x_5, x_5x_1\})$ with connectivity requirements $r(x_i, x_j) = 3$ for any $i, j \in \{2, 3, 4, 5\}$ and $r(x_1, x_j) = 0$ for any $j \in \{2, 3, 4, 5\}$. Choosing the degree-specification $m = 1$, the splitting-off at x_2, x_5 is admissible and $SLB(p') = SLB(p) - 1 = 3$.

Let $M_p = \max\{p(X) : X \subseteq V\}$. A set X with $p(X) = M_p$ will be called ***p*-maximal**. Clearly, if $M_p \leq 0$ then any splitting-off is admissible, so we will usually assume that $M_p > 0$. In this case if X and Y are two p-maximal sets then either both of $X \cap Y$ and $X \cup Y$ are p-maximal (if X and Y satisfy $(\cap\cup)$) or both of $X - Y$ and $Y - X$ are p-maximal (if X and Y satisfy $(-)$) by the positively skew-supermodularity of p.

We mention the following simple observation about **loop edges**. When we consider the degree-specified version of a covering problem, then usually we allow loop edges in the graph to be found. Therefore we usually use the terminology "a graph G satisfies the degree-specification m" to mean that $d^+_G(v) = m(v)$ for every $v \in V$ (note that $d^+_G(v)$ also counts the loop edges). However we could avoid this by the following transformation: if G satisfies the degree-specification m and contains a loop edge incident to some node u, but there exist another edge vw (possibly another loop but) not incident to u then deleting these two edges and introducing the edges uv and uw the obtained graph G' still satisfies the degree-specification m and $d_{G'}(X) \geq d_G(X)$ for any $X \subseteq V$, i.e. if G covers some function p then G' also covers p. Repeating this operation we can get rid of the loop edges, unless there exists a node $v \in V$ that has $m(v) > m(V - v)$. Therefore if we also made the assumption

$$m(v) \leq m(V - v) \text{ for every } v \in V \tag{2.8}$$

in our theorems, then we could avoid dealing with loop edges. However we will follow an approach that allows loop edges. Note that in the partition constrained problems in Chapter 5 the condition (2.8) is implicitly present among the others (and loop edges are not allowed in partition constrained problems).

2.3.1 Contraction of tight sets

A widely used ingredient of splitting-off algorithms is contraction of tight sets. This can be done since it does not change the admissibility structure (the set of admissible pairs does not change) and most of the proofs become simpler if tight sets are contracted. In this section we give the background of this simplification, however we will always try to avoid the use of contraction wherever the discussion does not become unnecessarily difficult without it.

If $T \subseteq V$ then contracting T roughly means that from now on we consider it to be a singleton. Formally this means that we define $V/T = V - T + v_T$ where v_T is a new node not in V. For any set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ we define $p/T : 2^{V/T} \to \mathbb{Z} \cup \{-\infty\}$ by $p/T(X) = p(X)$ if $v_T \notin X$ and $p/T(X) = p(X - v_T + T)$ if $v_T \in X$. For $m : V \to \mathbb{R}$ define $m/T : V/T \to \mathbb{R}$ with $m/T(v) = m(v)$ if $v \neq v_T$ and $m/T(v_T) = m(T)$: observe that regarding m to be a set function would give the same definition. In this contracted problem a splitting-off is admissible if it is admissible with respect to p/T. Note that p/T will inherit the interesting properties of p investigated in this paper (e.g. symmetry, (positively) crossing supermodularity, (positively) skew-supermodularity etc.). Contraction of a hypergraph $H = (V, \mathcal{E})$ is understood in the obvious way as $H/T = (V/T, \{e \in \mathcal{E} : T \cap e = \emptyset\} \cup \{e - T + v_T : e \in \mathcal{E}, T \cap e \neq \emptyset\})$, so we will usually not consider hyperedges with multiplicities (multihyperedges) after a contraction. However, for the graph of the edges split so far we must count the multiplicity in the loop edges obtained this way in order to satisfy the degree specification: this will not cause any confusion. One can check that $d_{H/T} = d_H/T$. The **contracted image v/T of a node $v \in V$** is defined as $v/T = v_T$, if $v \in T$, and $v/T = v$ otherwise. The main observation about contraction is the following.

Lemma 2.17. *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric, positively skew-supermodular function, $m \in C(p)$, and $u, v \in V$ with $m(u), m(v) > 0$. If we contract a p-positive tight set T then the splitting at u/T and v/T is admissible with respect to p/T if and only if the splitting at u and v is admissible with respect to p*

Proof. By the definition of p/T, if the splitting-off at u and v was admissible then it clearly stays admissible. Let us prove the other direction. Assume that $u/T, v/T$ becomes admissible while u, v was not admissible, i.e. there was a set $X \subseteq V$ with $p(X) \geq m(X) - 1$ with $u, v \in X$ (a dangerous set with respect to p: note that the inequality $p(X) \geq m(X) - 1$ implies that X is p-positive). Clearly, neither $T \subseteq X$ nor $X \cap T = \emptyset$ can hold. If $(\cap\cup)$ holds for X and T then $X \cup T$ is also dangerous, a contradiction. So $(-)$ must hold for them, meaning $X - T$ is also dangerous and $u, v \in X - T$, a contradiction again. $\square$

This lemma allows us to simplify some of the proofs by assuming that every tight set is a singleton. Note that if p is not only positively skew-supermodular but skew-supermodular, then we can also contract a tight set T with $p(T) = 0$.

Chapter 3

Covering skew-supermodular functions with hyperedges

This chapter is devoted to covering symmetric, positively skew-supermodular functions with hyperedges. The starting result is Theorem 3.2 of Szigeti. The application he had in mind was local edge-connectivity augmentation of hypergraphs by hyperedges of minimum total size. Here we generalize this theorem in many directions and we will consider other applications, too. The results of this section were found together with Tamás Király and they appeared in [12].

Recall that the hypergraph $H = (V, \mathcal{E})$ is said to **cover** the function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ if $d_H(X) \geq p(X)$ for every $X \subseteq V$. We will also use a different notion of covering: H is said to **weakly cover** the function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ if $b_H(X) \geq p(X)$ for every $X \subseteq V$, where

$$b_H(X) = |\{e \in \mathcal{E} : e \cap X \neq \emptyset\}|.$$

Note that b_H is submodular, monotone (but not symmetric), and

$$b_H(X) + b_H(Y) \geq b_H(X - Y) + b_H(Y - X) + |\{e \in \mathcal{E} : \emptyset \neq e \cap Y \subseteq X \cap Y\}|.$$

Notice that $b_H(v) = d_H^+(v)$ for a hypergraph H and a node v. The central problem of this chapter is the following.

Problem 3.1 (Covering a positively skew-supermodular function with hyperedges)**.** *Let us be given a symmetric, positively skew-supermodular function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ with a maximizing oracle. The* ***minimum version*** *of the problem is to find a hypergraph H covering p of minimum total size. The* ***degree-specified version*** *is to find a hypergraph H covering p that also satisfies a given degree-specification $m \in \mathbb{Z}_+^V$.*

In [42], Szigeti proved the following result, which solves the degree-specified version of the problem given above. It states that if the degree-specification is admissible then the required hypergraph exists.

Theorem 3.2 (Z. Szigeti, [42]). *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric, positively skew-supermodular set function, and $m : V \to \mathbb{Z}_+$ a degree specification.*

(i) *There exists a hypergraph H s.t. $d_H^+(v) = m(v)$ for every $v \in V$ and $d_H(X) \geq p(X)$ for every $X \subseteq V$ if and only if*

$$\sum_{v \in X} m(v) \geq p(X) \quad \textit{for every } X \subseteq V. \tag{3.1}$$

(ii) *Furthermore, if $m(v) \leq M_p = \max\{p(X) : X \subseteq V\}$ for any $v \in V$, then H can be chosen so that it consists of exactly M_p hyperedges.*

In this chapter we give different kinds of extensions of this theorem. The structure of this chapter is the following. First we show a simple proof of Szigeti's theorem with a different method compared to his: we use the operation of *merging hyperedges*. This already allows us to generalize his result slightly in one direction: in Theorem 3.3 we can tell when it is possible to obtain a hypergraph H_* covering the function p by merging hyperedges of a hypergraph H given in advance. In the next section first we show a generalization of Schrijver's supermodular colouring theorem, that is closely related to hypergraphs *weakly covering* a positively skew-supermodular function p. We show this relation in Theorem 3.9. Then we observe that under certain circumstances **weak covering already implies covering**. This together with the polyhedral observations on hypergraphs weakly covering the set function give extensions of Theorem 3.2: for example the optimal hypergraph can always be chosen to be nearly uniform. Furthermore, we obtain that under some circumstances we can solve the problem of covering two skew-supermodular functions simultaneously. In Section 3.4 we give applications of our results. We note that algorithmic aspects are not adressed in this chapter, we simply give minmax results here. The algorithm that we suggest finds the hyperedges of the augmenting hypergraph one by one, thus it is not polynomial. However, probably a similar argument as the one given in [29] would give polynomial algorithms for the problems discussed in this chapter.

3.1 Merging hyperedges

Let $H = (V, \mathcal{E})$ be a hypergraph. By *merging* two disjoint hyperedges of H we mean the operation of replacing them by their union. "Merging some hyperedges of H" means repeating this operation a few times. We note that part (i) of the following theorem due to Tamás Király already appeared in [30] and in Hungarian in [31].

Theorem 3.3. *Let $H = (V, \mathcal{E})$ be a hypergraph, and let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric, positively skew-supermodular set function with $k = M_p = \max\{p(X) : X \subseteq V\} \geq 0$, for which*

$$b_H(X) \geq p(X) \quad \textit{for every } X \subseteq V. \tag{3.2}$$

(i) *Then by merging some hyperedges of H we can obtain a hypergraph $H_* = (V, \mathcal{E}_*)$ that covers p.*

(ii) *Furthermore, if there are k hyperedges $f^1, f^2, \dots, f^k$ in H such that every hyperedge in $H - \{f^1, \dots, f^k\}$ is a singleton and $b_H(v) \leq k$ for any $v \in V$, then the merging operations can be organized in a way that $H_* = (V, \{f^1_*, f^2_* \dots, f^k_*\})$ where $f^i \subseteq f^i_*$ for every $i = 1, \dots, k$.*

Proof. We prove (i) by induction on the number of hyperedges of H (it is clearly true if $\mathcal{E} = \emptyset$). A set $X \subseteq V$ is called *tight* if $b_H(X) = p(X)$. By the properties of b_H and p, if X and Y are p-positive and tight, then either $X \cap Y$ and $X \cup Y$ are tight, or $X - Y$ and $Y - X$ are tight. Furthermore, if X and Y are p-positive and tight and there is a hyperedge e such that $\emptyset \neq e \cap Y \subseteq X \cap Y$, then $X \cap Y$ and $X \cup Y$ are tight.

Let e_0 be an arbitrary hyperedge of H. If there is no tight set X such that $e_0 \subseteq X$, then let $H' := H - e_0$ and $p' = p - d_{H_0}$ where $H_0 = (V, \{e_0\})$. The set function p' is symmetric and positively skew-supermodular, and $b_{H'}(X) \geq p'(X)$ for every $X \subseteq V$, so by induction there is a hypergraph H'_*, obtained by merging some hyperedges of H', such that $d_{H'_*}(X) \geq p'(X)$ for every $X \subseteq V$. It follows that $H_* := H'_* + e_0$ covers p. We can thus assume that there is a tight set X_0 such that $e_0 \subseteq X_0$: let X_0 be a maximal tight set containing e_0.

Suppose that there is no hyperedge $e \in \mathcal{E}$ such that $e \cap X_0 = \emptyset$. Then $p(V - X_0) = p(X_0) = b_H(X_0) > b_H(V - X_0)$ since $e_0 \subseteq X_0$, contradicting (3.2). Thus there is a hyperedge $e_1 \in \mathcal{E}$ such that $e_1 \cap X_0 = \emptyset$. Consider the hypergraph $H' := (V, \mathcal{E} - \{e_0, e_1\} + (e_0 \cup e_1))$, i.e. the hypergraph obtained by merging e_0 and e_1. If $b_{H'}(Y_0) < p(Y_0)$ for some $Y_0 \subseteq V$, then $e_0 \cap Y_0 \neq \emptyset$, $e_1 \cap Y_0 \neq \emptyset$, and Y_0 was tight. Since $\emptyset \neq e_0 \cap Y_0 \subseteq X_0 \cap Y_0$, $X_0 \cup Y_0$ is also tight, which contradicts the maximality of X_0.

We proved that H' and p satisfy (3.2), so by induction there is a hypergraph H_*, obtained by merging some hyperedges of H' (hence obtained by merging some hyperedges of H), that covers p.

The proof of (ii) is similar to the proof of Theorem 3.2 by Szigeti. We will use the following observation. Let $X, Y \subseteq V$ such that X is tight and $p(Y) = k$. If $(\cap\cup)$ applies for X and Y then $p(X \cup Y) \leq p(Y) = k$ implies that $p(X \cap Y) \geq p(X) = b_H(X) \geq b_H(X \cap Y) \geq p(X \cap Y)$, so every inequality is satisfied with equality here (including $p(X \cup Y) = k$). On the other hand, if $(-)$ applies for X and Y then $p(Y - X) \leq p(Y) = k$ implies that $p(X - Y) \geq p(X) = b_H(X) \geq b_H(X - Y) \geq p(X - Y)$, so every inequality is satisfied with equality here (including $p(Y - X) = k$).

We will prove the statement indirectly: suppose that H, p and $f^1, \ldots, f^k$ form a counterexample with k as small as possible and, subject to that, $|V - f^k|$ as small as possible. Trivially, $k > 0$. Suppose that there is a set Y with $p(Y) = k$ that is disjoint from f^k. Since $b_H(Y) \geq p(Y) = k$ there must be a hyperedge $e \in \mathcal{E} - \{f^1, \ldots, f^{k-1}, f^k\}$ that intersects Y: since these hyperedges are singletons, in fact $e \subseteq Y$. Let H' be obtained from H by merging f^k and e into a hyperedge f'^k. We claim that H' does not violate (3.2): if it does then there was a tight set X such that $f^k \cap X \neq \emptyset$, $e \cap X \neq \emptyset$ ($e \subseteq X$ in fact). But $b_H(X) > b(X \cap Y)$ because of the edge f^k, implying that $(\cap\cup)$ cannot apply for X and Y. On the other hand, $b_H(X) > b_H(X - Y)$ because of the hyperedge e, so $(-)$ cannot apply for X and Y either, a contradiction. By the minimal choice of H, the statement is true for H', but then also for H, a contradiction. So in our minimal counterexample f^k intersects every set Y with $p(Y) = k$.

Similarly we claim that in this minimal counterexample f^k must cover every vertex v with $b_H(v) = k$. Assume that this is not the case and v is such a vertex not covered by f^k. Then there must be a hyperedge $e \in \mathcal{E} - \{f^1, \ldots, f^{k-1}, f^k\}$ that covers v and if we merge f^k with e then the hypergraph H' obtained will not violate (3.2), since every set Y that intersects both e and f^k has $b_H(Y) \geq k+1$, so it cannot be tight. Therefore the statement is true for H', and then also for H, a contradiction.

We claim that there is no tight set X satisfying $f^k \subseteq X$. Assume that this is not true and let X be such a tight set. Let Y be an arbitrary set with $p(Y) = k$ (such a set exists by the definition of k). If $(\cap\cup)$ applies for X and Y then $p(X \cup Y) = k$, but then $p(V - (X \cup Y)) = k$, too, but this set is not covered by f^k. However, $(-)$ cannot apply for X and Y either, because then $Y - X$ would be a set with $p(Y - X) = k$ not covered by f^k. So we really obtained that there is no tight set containing f^k.

Let $H' = H - f^k$ and $p' = p - d_{H^k}$, where $H^k = (V, \{f^k\})$. Then $\max\{p'(X) : X \subseteq V\} = k - 1$, $b_{H'}(v) \leq k - 1$ for every $v \in V$, and $b_{H'} \geq p'$, thus H', p' and $f^1, \ldots, f^{k-1}$ must satisfy the statement to be proved (otherwise H was not a minimal counterexample), so there exists a hypergraph $H'_* = (V, \{f^1_*, \ldots, f^{k-1}_*\})$ that covers p' and satisfies $f^i \subseteq f^i_*$ for any i between 1 and $k - 1$. But then one can easily check that $H_* = H'_* + f^k$ satisfies our requirements, so H was not a counterexample. $\square$

Theorem 3.2 corresponds to the case when H consists of hyperedges of size 1, and $m(v)$

is the multiplicity of $\{v\}$ in H. We mention that the assumptions of Theorem 3.3 (ii) are really needed: if a hypergraph H weakly covers a positively skew-supermodular function p and H has more than $k = M_p$ nonsingleton hyperedges then we cannot necessarily merge hyperedges of H with maintaining the inequality $b_H \geq p$ (but by Theorem 3.3 (i), only if H in fact covers p). A simple example above the 3 element ground set $V = \{v_1, v_2, v_3\}$ is the following: let $p(X) = 2$ for any nonempty $X \subsetneq V$ and $p(\emptyset) = p(V) = 0$ and let $H = (V, \{v_1v_2, v_2v_3, v_3v_1\})$.

3.2 Weak covering of positively skew-supermodular functions

In this section we first show a generalization of Schrijver's supermodular colouring theorem. We want to describe this in the context of hypergraphs instead of colourings: let us show the connection. A *k-colouring* is a partition $X_1, \ldots, X_k$ of V (where some of these classes may even be empty). If a set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is also given then a k-colouring is *good* (for p) if $|\{i : X_i \cap X \neq \emptyset\}| \geq p(X)$ for any $X \subseteq V$. Observe that the colouring $X_1, \ldots, X_k$ is good if and only if the hypergraph with edge set $X_1, \ldots, X_k$ weakly covers the set function p.

Let us state the important results on supermodular colourings. The following theorem is called the **supermodular colouring theorem**.

Theorem 3.4 (Schrijver, [39]). *Let $p_1, p_2 : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be intersecting supermodular functions and let k be a positive integer. There exists a colouring $X_1, X_2, \ldots, X_k$ of V that is good for both p_1 and p_2 if and only if*

$$p_i(X) \leq \min\{k, |X|\} \text{ for any } X \subseteq V \text{ and } i = 1, 2.$$

A nice and simple proof of this theorem was given by Éva Tardos (a further simplified form of this proof can be found in Schrijver's book [41], pages 849-851). The key of her proof is the following lemma.

Lemma 3.5 (Tardos, [44]). *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be an intersecting supermodular function and let k be a positive integer. Assume that $p(X) \leq \min\{k, |X|\}$ for any $X \subseteq V$. Then the polyhedron*

$$Q = \{x \in \mathbb{R}^V : x(Z) \geq 1 \text{ if } p(Z) = k, x(Z) \leq |Z| - p(Z) + 1 \ \forall Z \subseteq V,\ 0 \leq x \leq 1\}$$

is a nonempty integer g-polymatroid.

To prove Theorem 3.4 it is enough to show that the integer elements of the polyhedron defined in Lemma 3.5 correspond to the characteristic vectors of the possible colour classes of good colourings: we will see exact details in a more general setting later. Szigeti showed that Theorem 3.2 implies a special case of the supermodular colouring theorem. In this section we develop tools to prove a reverse implication. First we show that the supermodular colouring theorem is true in a more general form, namely for positively skew-supermodular functions instead of intersecting supermodular functions – this was observed by Tamás Király in [28]. Then we show that **for symmetric, positively skew-supermodular set functions weak covering implies covering if the number of hyperedges equals the maximum value of the set function**. This enables us to derive generalizations of Szigeti's theorem.

Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be an arbitrary positively skew-supermodular function and let $C = C(p)$ be the contrapolymatroid defined in (1.11). Let $k \geq \max\{p(X) : X \subseteq V\}$ and let $y \leq (k, k, \dots, k)$ be an integer vector in C (note that the vector $(k, k, \dots, k)$ is in C, so such a y exists). Our aim is to find a hypergraph of k hyperedges satisfying the degree-specification y that weakly covers p. Note that in the following theorems the symmetry of p is not necessarily assumed.

Let us define the polyhedron

$$Q = Q(p, k, y) = \{x \in \mathbb{R}^V : x(Z) \geq 1 \text{ if } p(Z) = k,$$
$$x(Z) \leq y(Z) - p(Z) + 1 \ \forall Z \subseteq V,\ 0 \leq x \leq y\}.$$

One can observe that $y/k \in Q$, so Q is non-empty.

The proof of the following lemma essentially follows the line of the proof of Lemma 3.5 that appears in Schrijver's book [41].

Lemma 3.6. *Q is a (nonempty, integer) g-polymatroid.*

Proof. We will show that $Q' = y - Q$ is a g-polymatroid, from which the statement follows. One can see that

$$Q' = \{x \in \mathbb{R}^V :\ x(Z) \geq p(Z) - 1 \ \forall Z \subseteq V, x(T) \leq y(T) - 1 \text{ if } p(T) = k,\ 0 \leq x \leq y\}.$$

Let

$$\mathcal{D} = \{T \subseteq V : p(T) = k \text{ but } p(Y) < k \text{ for any } Y \subsetneq T\},$$

and

$$\mathcal{C} = \{X \subseteq V : X \subseteq T \text{ for some } T \in \mathcal{D} \text{ or } X \cap T = \emptyset \ \forall T \in \mathcal{D}\}.$$

The positively skew-supermodularity of p implies that $\mathcal{D}$ is a subpartition. It is also clear that

$$Q' = \{x \in \mathbb{R}^V :\ x(Z) \geq p(Z) - 1\ \forall Z \subseteq V, x(T) \leq y(T) - 1\ \forall T \in \mathcal{D},\ 0 \leq x \leq y\}.$$

We claim that Q' is actually equal to

$$Q'' = \{x \in \mathbb{R}^V :\ x(Z) \geq p(Z) - 1\ \forall Z \in \mathcal{C}, x(T) \leq y(T) - 1\ \forall T \in \mathcal{D},\ 0 \leq x \leq y\}.$$

We only need to show that $Q'' \subseteq Q'$. Let $x \in Q''$ and $Z \subseteq V$ be arbitrary: we have to show that $x(Z) \geq p(Z) - 1$. To prove this assume that Z intersects $t > 0$ members of $\mathcal{D}$ and prove by induction on t. Let $T \in \mathcal{D}$ be one of the t members of $\mathcal{D}$ intersected by Z.

Assume T and Z satisfy $(\cap\cup)$. This implies $p(Z) \leq p(Z \cap T)$ (since $p(T)$ is maximum). Then $x(Z) \geq x(Z \cap T) \geq p(Z \cap T) - 1 \geq p(Z) - 1$, as claimed. Otherwise T and Z satisfy $(-)$, implying $p(Z) \leq p(Z - T)$. Then $x(Z) \geq x(Z - T) \geq p(Z - T) - 1 \geq p(Z) - 1$, since $Z - T$ intersects $t - 1$ members of $\mathcal{D}$, so we can use induction.

Let $f(X) = \max\{0, \sum_{i=1}^{t}(p(X_i) - 1)\colon\ X_1, \ldots, X_t$ is a subpartition of $X\}$ for any $X \in \mathcal{C}$ and $-\infty$ otherwise, and let $g(X) = y(X) - 1$ for any $X \in \mathcal{D}$ and ∞ otherwise. The set function g is intersecting submodular because it has finite values on a subpartition, and it clearly satisfies the cross inequality (1.9) with f for any properly intersecting pair X, Y. To see that f is intersecting supermodular, let U be a maximal member of $\mathcal{C}$, and let f_U and p'_U be the set functions f and $p(X) - 1$ restricted to the subsets of U. Then p'_U is skew-supermodular and f_U is the supermodular function defining $C(p'_U)$ (see Theorem 1.17). Since the maximal members of $\mathcal{C}$ form a partition, f is intersecting supermodular on V.

We can conclude that f and g form a weak pair, so $Q(f, g)$ is a g-polymatroid by Theorem 1.15. Furthermore, by Theorem 1.16, $Q' = Q'' = Q(f, g) \cap \{x :\ 0 \leq x \leq y\}$ is also a g-polymatroid, since it is the intersection of a g-polymatroid with a box. □

The integer points of Q are closely related to multihyperedges of a multihypergraph that weakly covers p. More precisely, the following statement can be proved with a similar proof to that of Theorem 3.9. Let $Q^* = Q \cap \{x \in \mathbb{R}^V :\ x(v) = 1$ if $y(v) = k\}$: by Theorem 1.16, Q^* is an integer g-polymatroid, too (note that $d_H^+(v)$ is not necessarily equal to $b_H(v)$ for multihypergraphs).

Theorem 3.7. *An integer vector is in Q^* if and only if it is a multihyperedge of a multihypergraph $H = (V, \mathcal{E})$ containing k multihyperedges, which weakly covers p and satisfies $d_H^+(v) = y(v)$ for any $v \in V$.* □

However, we would like to speak of hypergraphs instead of multihypergraphs. Thus we would like to prove a version of Theorem 3.7 that applies to hypergraphs instead of multihypergraphs. We can do this, moreover, we can prove a similar statement about nearly uniform hypergraphs. To this end, let us modify Q slightly. By the basic properties of g-polymatroids we have the following.

Corollary 3.8. *The polyhedron*

$$Q_1 = Q_1(p,k,y) = Q^* \cap \left\{ x \in \mathbb{R}^V : \ 0 \le x \le 1, \ \left\lfloor \frac{y(V)}{k} \right\rfloor \le x(V) \le \left\lceil \frac{y(V)}{k} \right\rceil \right\} \tag{3.3}$$

is a non-empty integer g-polymatroid.

Proof. The statement is implied by Theorem 1.16. The non-emptiness follows from the fact that $\frac{y}{k} \in Q_1$ since $y(v) \le k$ for every $v \in V$. □

Now we show that Q_1 is exactly the convex hull of the characteristic vectors of hyperedges that can appear in the type of hypergraphs we are looking for. To avoid some trivial degenerate cases we assume that $y(V) \ge k$.

Theorem 3.9. *An integer vector $x \in \mathbb{Z}^V$ is in Q_1 if and only if it is the characteristic vector of a hyperedge of a nearly uniform hypergraph $H = (V, \mathcal{E})$ containing k hyperedges, which weakly covers p and satisfies $b_H(v) = y(v)$ for any $v \in V$.*

Proof. If $H = (V, \mathcal{E})$ is a nearly uniform hypergraph containing k hyperedges that weakly covers p and satisfies $b_H(v) = y(v)$ for every $v \in V$, then clearly $\chi_e \in Q_1 \cap \mathbb{Z}^V$ for any $e \in \mathcal{E}$.

Let $x \in Q_1 \cap \mathbb{Z}^V$. We need to prove that there is a hypergraph H with the desired properties. We prove by induction on k. Let $H_k = (V, \{e_k\})$ where $\chi_{e_k} = x$. If $k = 1$, then H_k satisfies the conditions. Otherwise, let $p^* = p - b_{H_k}$, and let $y^* = y - x$. The set function p^* is positively skew-supermodular. By the choice of x we have $\max\{p^*(X) : X \subseteq V\} \le k - 1$, $y^* \in C(p^*)$, $y^*(v) \le k - 1$ for every $v \in V$, and $y^*(V) \ge k - 1$. Let $Q_1^* = Q_1(p^*, k-1, y^*)$. By Corollary 3.8, Q_1^* is also a non-empty g-polymatroid; let x^* be an arbitrary integer element of Q_1^*. By induction there is a nearly uniform hypergraph H^* with $k - 1$ hyperedges (one of them with characteristic vector x^*) satisfying $b_{H^*} \ge p^*$ and $b_{H^*}(v) = y^*(v)$ for every $v \in V$. Thus $H = H^* + H_k$ weakly covers p and $b_H(v) = y(v)$ for every $v \in V$. In addition, since H^* is nearly uniform and $\lfloor y(V)/k \rfloor \le y^*(V)/(k-1) \le \lceil y(V)/k \rceil$ H, it follows that H is nearly uniform, too. □

The above theorem of course implies the following.

Corollary 3.10. *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a positively skew-supermodular function, let $k \geq \max\{p(X) : X \subseteq V\}$ be an integer, and let $y \in C(p)$ be an integer vector with $y(V) \geq k$ and $y(v) \leq k$ for every $v \in V$. Then there is a nearly uniform hypergraph containing k hyperedges that weakly covers p and satisfies $b_H(v) = y(v)$ for every $v \in V$.*

Next, we prove that for **symmetric** set functions we can strengthen the above result by replacing "weakly covers" by "covers", provided that $k = M_p = \max\{p(X) : X \subseteq V\}$.

Lemma 3.11. *If $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is a **symmetric** positively skew-supermodular function, $k = \max\{p(X) : X \subseteq V\}$, and H is a hypergraph containing exactly k hyperedges, then $b_H \geq p$ implies that H **covers** p.*

Proof. The simplest way of proving this at this point is just to say that it obviously follows from Theorem 3.3 (i). However we give a direct proof, too. Suppose that H does not cover p, so there is a set X with $b_H(X) \geq p(X) > d_H(X) = b_H(X) - i_H(X)$, where $i_H(X)$ denotes the number of hyperedges of H *induced* by X. By the assumptions there is a set T with $p(T) = k$. Since H contains exactly k hyperedges, $p(X \cup T) = p(V - (X \cup T)) \leq k - i_H(X)$ and $p(T - X) \leq k - i_H(X)$ also follows. If $(\cap\cup)$ applies for X and T then $p(X \cap T) \geq p(X) + i_H(X) > b_H(X) \geq b_H(X \cap T)$, and if $(-)$ applies for X and T then $p(X - T) \geq p(X) + i_H(X) > b_H(X) \geq b_H(X - T)$, both of which contradicts our assumptions. □

This immediately implies the following generalization of Theorem 3.2.

Theorem 3.12. *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric, positively skew-supermodular function, let $k = \max\{p(X) : X \subseteq V\}$, and let $y \in C(p)$ be an integer vector with $y(v) \leq k$ for every $v \in V$. Then there is a nearly uniform hypergraph containing k hyperedges that covers p and satisfies $b_H(v) = y(v)$ for every $v \in V$.*

Note that if we want to find a hypergraph of minimum total size that covers p, it suffices to find a vector $y \in C(p)$ with $y(V)$ minimal. Such a vector automatically satisfies $y(v) \leq k$ for every $v \in V$ because the vector defined by $y'(v) = \min\{k, y(v)\}$ is also in $C(p)$. We can even go a bit further: instead of minimizing the total size of H, we may want to minimize $\sum_{v\in V} c(v) \cdot b_H(v)$, where c is a non-negative cost function on the nodes. In this case we should find a vector $y \in C(p)$ with $\sum_{v\in V} c(v)y(v)$ minimal.

3.3 Covering two positively skew-supermodular functions

As in the case of Schrijver's supermodular colouring theorem, these results can be generalized using the fact that the intersection of two integer g-polymatroids is an integer polyhedron (a special type of submodular flow polyhedron).

Let $p_1, p_2 : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be two positively skew-supermodular functions, and let $k \geq \max\{p_i(X) : \ X \subseteq V, \ i = 1, 2\}$. Let $y \in C(p_1) \cap C(p_2) \cap \mathbb{Z}^V$ such that $y(v) \leq k$ for every $v \in V$ and $y(V) \geq k$ (for example $(k, k, \dots, k)$ is such a vector). Let

$$R = R(p_1, p_2, k, y) := Q_1(p_1, k, y) \cap Q_1(p_2, k, y),$$

where Q_1 is defined as in (3.3). Then $y/k \in R$, and R is the intersection of two integer g-polymatroids, so R is a non-empty integer polyhedron.

Theorem 3.13. *An integer vector $x \in \mathbb{Z}^V$ is in R if and only if it is the characteristic vector of a hyperedge of a nearly uniform hypergraph $H = (V, E)$ containing k hyperedges which weakly covers both p_1 and p_2 and satisfies $b_H(v) = y(v)$ for every $v \in V$.*

Proof. The proof is analogous to the proof of Theorem 3.9; the only difference is that we have to define p_1^* and p_2^*, and use induction on $R(p_1^*, p_2^*, k-1, y^*)$. □

The results above imply the following "skew-supermodular colouring theorem" which is an extension of Schrijver's supermodular colouring theorem. It follows from Theorem 3.13 by choosing $y = (1, 1, \dots, 1)^T \in \mathbb{R}^V$.

Theorem 3.14 (T. Király, [28])**.** *Let $p_1, p_2 : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be two positively skew-supermodular functions and $k \geq 1$ an integer. Then there is a k-colouring that is good for both p_1 and p_2 if and only if*

$$p_i(X) \leq \min\{k, |X|\} \text{ for any } X \subseteq V \text{ and } i = 1, 2.$$

Using Lemma 3.11 we obtain the following theorem.

Theorem 3.15. *Let $p_1 : 2^V \to \mathbb{Z} \cup \{-\infty\}$ and $p_2 : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be two symmetric, positively skew-supermodular set functions such that*

$$\max\{p_1(X) : X \subseteq V\} = \max\{p_2(X) : X \subseteq V\} = k, \tag{3.4}$$

and let $y \in C(p_1) \cap C(p_2)$ be an integer vector with $y(v) \leq k$ for every $v \in V$. Then there is a nearly uniform hypergraph containing k hyperedges that covers both p_1 and p_2 and satisfies $b_H(v) = y(v)$ for every $v \in V$.

This theorem can be used to find a hypergraph of minimum total size that covers both p_1 and p_2 provided that (3.4) holds. By the argument presented at the end of the previous section, it suffices to find a vector $y \in C(p_1) \cap C(p_2)$ with $y(V)$ minimal, since such a vector automatically satisfies $y(v) \leq k$ for every $v \in V$. We can also minimize $\sum_{v \in V} c(v) \cdot b_H(v)$, where c is a non-negative cost function on the nodes.

We will show in Section 3.4.1 that without the assumption that the maximum values of the two set functions are the same, the problem of minimizing the total size of H becomes NP-hard.

3.4 Applications

In this section we present applications of our results. Besides the local edge-connectivity augmentation of hypergraphs discussed by Szigeti in [42], we show two other applications.

3.4.1 Local edge-connectivity augmentation of hypergraphs

The **local edge-connectivity augmentation of hypergraphs with hyperedges of minimum total size** (solved by Szigeti in [42]) is the following problem.

Problem 3.16. *Given a hypergraph $H_0 = (V, \mathcal{E}_0)$ and a symmetric edge-connectivity requirement $r : V \times V \to \mathbb{Z}_+$, find a hypergraph H of minimum total size such that $H_0 + H$ is r-edge-connected, meaning that*

$$\lambda_{H_0+H}(u, v) \geq r(u, v) \textit{ for every } u, v \in V. \tag{3.5}$$

Let us define the set function R_{loc} with (2.5). As we have seen in Section 2.1.4, a hypergraph H satisfies (3.5) if and only if H covers $p = R_{loc} - d_{H_0}$. Since p is a positively skew-supermodular function, applying Theorem 3.12 gives the following extension of Szigeti's result.

Theorem 3.17. *The optimal solution of the local edge-connectivity augmentation of hypergraphs with hyperedges of minimum total size can be chosen to be nearly uniform.*

Now we use Theorem 3.15 to get an even more general result. Consider the following problem, the **simultaneous local edge-connectivity augmentation problem**. Given two hypergraphs H_1, H_2 on the same ground set V, two symmetric requirement functions $r_1, r_2 : V \times V \to \mathbb{Z}$, and a nonnegative cost function $c : V \to \mathbb{R}_+$, we want to find a hypergraph H of minimum total cost such that $H_i + H$ is r_i-edge-connected for $i = 1, 2$ (the cost of H is $\sum_{v \in V} c(v) b_H(v)$).

Theorem 3.18. *Assume that* $\max\{r_1(u, v) - \lambda_{H_1}(u, v) : u, v \in V\} = \max\{r_2(u, v) - \lambda_{H_2}(u, v) : u, v \in V\}$*, in other words $M_{p_1} = M_{p_2}$, where p_1 and p_2 are the deifiency functions of the two instances. If $y \in C(p_1) \cap C(p_2) \cap \mathbb{Z}^V$ then there exists a nearly uniform hypergraph $H = (V, \mathcal{E})$ satisfying the degree-specification y that solves the simultaneous local edge-connectivity augmentation problem (i.e. covers both p_1 and p_2).*

The following theorem shows that without the assumption on the maximum deficiencies the problem becomes NP-complete: the reduction is similar to that used in [16].

Theorem 3.19. *The simultaneous local edge-connectivity augmentation problem of two hypergraphs is in general NP-complete, even if the cost function is constant.*

Proof. The problem is clearly in NP. To show its completeness consider the **Special Bin-Packing Problem (SBP)**. An instance of this problem consists of a set of positive integers $W = \{w_1, w_2, \dots, w_n\}$ (**weights**), a set of positive integers $B = \{b_1, b_2, \dots, b_m\}$ (**bins**) such that $\gamma = \sum_{i=1}^{n} w_i = \sum_{j=1}^{m} b_j$. The SBP problem asks whether there exists a partition $W_1, W_2 \dots, W_m$ of W such that $\sum_{w \in W_j} w = b_j$ for every $j = 1, 2, \dots, m$. This problem is shown to be strongly NP-complete in [15], i.e. it remains NP-complete even if the weights and bins are unary encoded. We will reduce the unary-encoded SBP problem to our problem. For each weight $w_i \in W$ consider a set X_i such that $|X_i| = w_i$ and similarly, for each bin $b_j \in B$ let Y_j be such that $|Y_j| = b_j$. The sets $X_i\,(i = 1, 2, \dots, n)$ and $Y_j\,(j = 1, 2, \dots, m)$ are assumed to be pairwise disjoint. Let $X = \cup_{i=1}^{n} X_i$ and $Y = \cup_{j=1}^{m} Y_j$. The ground set of the two hypergraphs is $V = X \cup Y$. The edge-set of H_1 is the union of a complete graph on X and a complete graph on Y. The requirement function r_1 is uniformly γ. The edge-set of H_2 consists of hyperedges Y_j for every $j = 1, 2, \dots, m$ and the requirement is $r_2(u, v) = 1$ if $u, v \in X_i$ for some i, and 0 otherwise. One can check that there is a hypergraph H of total size at most 2γ such that $H_i + H$ is r_i-edge-connected for $i = 1, 2$ if and only if the SBP problem is solvable: the hypergraph H must be in fact a graph, more precisely a complete matching between X and Y (note that in H_1 the degree of any node is $\gamma - 1$). The details are left to the reader. $\square$

3.4.2 The node-to-area connectivity augmentation problem in hypergraphs

The node-to-area connectivity augmentation problem for graphs was solved by Ishii and Hagiwara [26]. Here we consider a hypergraphic version of the problem. Note that this is not a generalization of the node-to-area connectivity augmentation of graphs.

Problem 3.20 (Node-to-area connectivity augmentation problem in hypergraphs)**.** *Given a hypergraph $H_0 = (V, \mathcal{E}_0)$, a collection of subsets $\mathcal{W}$ of V and a function $r : \mathcal{W} \to \mathbb{Z}_+$, our aim is to find a hypergraph H of minimum total size such that*

$$\lambda_{H_0+H}(x, W) \geq r(W) \text{ for any } W \in \mathcal{W} \text{ and } x \in V. \tag{3.6}$$

Define the set function R_{N2A} by (2.4) and let $q = R_{N2A} - d_{H_0}$. The function q is intersecting negamodular, so $p = q^s$ is positively skew-supermodular. From Theorems 3.12 and 3.15 we get the following.

Theorem 3.21. *The optimal solution of the node-to-area connectivity augmentation problem in hypergraphs can be chosen to be nearly uniform. If we have two instances of the* *problem (on the same node set) such that for the corresponding requirement functions* p_1 *and* p_2 *we have* $\max\{p_1(X): X \subseteq V\} = \max\{p_2(X): X \subseteq V\}$ *and* $y \in C(p_1) \cap C(p_2) \cap \mathbb{Z}^V$ *then there exists a nearly uniform hypergraph* $H = (V, \mathcal{E})$ *satisfying the degree-specification* y *that solves the simultaneous node-to-area edge-connectivity augmentation problem (i.e. covers both* p_1 *and* p_2*).*

Note that the node-to-area connectivity augmentation problem for graphs was solved in [26] for the case when the set function R_{N2A} does not take the value 1; it was shown that without this assumption the problem is NP-hard. By the above theorem, this assumption is not needed in the hypergraphic case.

3.4.3 Augmenting the global arc-connectivity of mixed hypergraphs

This problem is a hypergraphic version of the global arc-connectivity augmentation of mixed graphs with undirected edges solved by Bang-Jensen, Jackson and Frank [2]. The problem is the following.

Problem 3.22 (Augmenting the global arc-connectivity of mixed hypergraphs)**.** *Let* $M = (V, \mathcal{A})$ *be a mixed hypergraph,* $r \in V$ *is a designated root node, and* k, l *be nonnegative integers. Find a hypergraph* H *of minimum total size such that* $M + H$ *is* (k, l)*-arc-connected from root* r*.*

Since this problem is a special case of the problem of covering a positively skew-supermodular function with hyperedges, it was also solved by Szigeti. Our results imply the following generalizations.

Theorem 3.23. *If* $M = (V, \mathcal{A})$ *is a mixed hypergraph,* $r \in V$ *is a designated root node,* k, l *are nonnegative integers, and* $y \in \mathbb{Z}^V$ *is a degree-specification then there exists a hypergraph* H *such that* $M + H$ *is* (k, l)*-arc-connected from* r *and* $d_H^+(v) = y(v)$ *for every* $v \in V$ *if and only if* $y \in C(p)$ *with* $p = q^s_{M,r,k,l}$*. Furthermore,* H *can be chosen to be nearly uniform.*

Theorem 3.24 (Simultaneous global arc-connectivity augmentation of mixed hypergraphs)**.** *If we have two instances of Problem 3.22 given by* (M_1, r_1, k_1, l_1) *and* (M_2, r_2, k_2, l_2) *on a common node set* V*, and for the set functions* $q_1 = q_{M_1,r_1,k_1,l_1}$*,* $q_2 = q_{M_2,r_2,k_2,l_2}$ *we have*

$$\max\{q_1(X): X \subseteq V\} = \max\{q_2(X): X \subseteq V\},$$

then there exists a nearly uniform hypergraph $H = (V, \mathcal{E})$ *satisfying the degree-specification* y *that solves the simultaneous local edge-connectivity augmentation problem (i.e. covers both* q_1 *and* q_2*) if and only if* $y \in C(q_1^s) \cap C(q_2^s) \cap \mathbb{Z}^V$*. Furthermore,* H *can be chosen to be nearly uniform.*

Chapter 4

Covering skew-supermodular functions with graph edges: a new approach to splitting-off

In this chapter we mainly consider the problem of covering our function only with graph edges. However our approach gives rise to a different question, too, which is the following: augment the edge-connectivity of a (mixed or undirected) hypergraph without increasing its rank. This question naturally arose from our approach to splitting-off, as it will be clear shortly. We will call it the **rank respecting augmentation problem**: exact definitions will be given later. Nevertheless, the main concern of this chapter is the following problem.

Problem 4.1 (Covering a positively skew-supermodular function with graph edges)**.** *Let us be given a symmetric, positively skew-supermodular function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ with a maximizing oracle. The* ***minimum version*** *of the problem is to find a graph G covering p with a minimum number of edges. The* ***degree-specified version*** *is to find a a graph G covering p that also satisfies a given degree-specification $m \in \mathbb{Z}_+^V$.*

We remind the reader that this problem is in general NP-complete, even for special skew-supermodular functions. Let us mention two cases that are already NP-complete. It was shown in [16] that the *local edge-connectivity augmentation of a hypergraph with a minimum number of graph edges* (introduced in Section 2.1.4) is NP-complete. The proof given there shows that the *local edge-connectivity augmentation of a hypergraph with a degree-specified graph* is also NP-complete. The other NP-complete special case is the *general node-to-area connectivity augmentation problem in graphs* (see Theorem 2.7).

The results presented in this chapter appeared in our joint paper with Tamás Király [11], except for those in Section 4.4.3 which appeared in [7].

4.1 Previous results - brief history

As we have mentioned in the previous section, the problem of covering a symmetric skew-supermodular function with a minimum number of graph edges is in general NP-complete. However, many special cases can be solved in polynomial time: let us review some of these cases.

Global edge-connectivity augmentation of graphs was solved by Watanabe and Nakamura [45]. However, their algorithm is not strongly polynomial. Frank [19] gave the first strongly polynomial algorithm for this problem. These results will be stated in Chapter 5, since they fall in the class of *covering a symmetric crossing supermodular function.*

Frank also proved in [19] that the local edge-connectivity augmentation of graphs can also be solved in strongly polynomial time. His proof uses the **splitting-off** technique to solve the degree specified problem. The solution of the local edge-connectivity augmentation problem relies on the classical splitting lemma of Mader:

Lemma 4.2 (Mader's lemma). *Let $G = (V + s, E)$ be such that there is no cut edge incident to s and $d_G(s) > 3$. Then there exists a splitting-off at s that preserves the local edge-connectivities in V.*

Frank discovered the relationship between the degree-specified and the minimum versions of augmentation problems that we sketched in Section 2.2. He also observed that Mader's lemma cited above is essentially equivalent to the degree-specified local edge-connectivity augmentation of graphs. Thus he gave the solution of the problem that is the following.

Theorem 4.3 (A. Frank, [19]). *Let G_0 be a graph and let $r : V \times V \to \mathbb{Z}_+$ be a symmetric edge-connectivity requirement such that $r(u, v) \neq 1$ for any pair $u, v \in V$. There exists a graph G satisfying the degree specification $m \in \mathbb{Z}_+^V$ such that $G_0 + G$ is r-edge-connected if and only if $m(V)$ is even and*

$$m(X) \geq R(X) - d_{G_0}(X) \text{ for any } X \subseteq V,$$

where R_{loc} is defined with (2.5). The minimum number of graph edges that make G_0 r-edge-connected is equal to

$$\max\{\lceil 1/2 \sum_{X \in \mathcal{X}} (R_{loc}(X) - d_{G_0}(X))\rceil : \mathcal{X} \text{ is a subpartition of } V\}.$$

Note that Frank actually solved the more general problem without the assumption that $r(u, v) \neq 1$, but the answer is a little more technical if this can also happen.

The next result is about augmenting the arc-connectivity of a mixed graph with undirected edges. For simplicity, we only state the minimum version.

Theorem 4.4 (J. Bang-Jensen, A. Frank, and B. Jackson, [2]). *Let $M_0 = (V, E_0)$ be a mixed graph and $k \geq 2$ integer. The minimum number of (undirected) graph edges that make M_0 k-arc-connected is equal to*

$$\max\{\lceil 1/2 \sum_{X \in \mathcal{X}} k - \beta_{M_0}(X)) \rceil : \mathcal{X} \textit{ is a subpartition of } V\},$$

where $\beta_{M_0}(X) = \min\{\varrho_{M_0}(X), \delta_{M_0}(X)\}$ for any $X \subseteq V$.

Note that the above theorems state that solution of the minimum version only depends on the bound *SLB*, and the solution of the degree-specified version exists if and only if the degree specification m is admissible (and $m(V)$ is even).

In the following problem this is not exactly the case: the optimal solution of the minimum version can be one bigger than the bound $\lceil \frac{1}{2} SLB \rceil$ obtained from subpartitions. This problem is the node-to-area connectivity augmentation problem solved by Ishii and Hagiwara.

Theorem 4.5 (Ishii and Hagiwara [26]). *Let a node-to-area connectivity augmentation problem be given by the graph $G_0 = (V, E_0)$, a collection of subsets $\mathcal{W}$ of V and a function $r : \mathcal{W} \to \mathbb{Z}_+ \setminus \{1\}$. The minimum number of graph edges $|F|$ such that $G_0 + F$ satisfies the area requirements is equal to $\lceil \frac{1}{2} SLB(p) \rceil$, unless there is a certain configuration called $\mathcal{W}$-configuration present, in which case the optimum is $\lceil \frac{1}{2} SLB(p) \rceil + 1$, where $p = R^s_{N2A} - d_{G_0}$ and R_{N2A} is defined by (2.3).*

The definition of a $\mathcal{W}$-configuration is defined in [26] where it is called *P-property*. In Section 4.3.2 we will give a simple proof of the fact that the optimum is at most $\lceil \frac{1}{2} SLB(p) \rceil + 1$.

4.2 A simple lemma and algorithm

In the results below we will use the *splitting-off technique*. The basics of this technique are described in Section 2.3. The starting point of our results is Lemma 4.6. This lemma was also found by Nutov who sketched a proof in [36]. For the special case when p is obtained from local edge-connectivity augmentation requirements in a hypergraph, this lemma was implicitly also shown by Ben Cosh in [15]. However the proof presented here is simpler than the previous ones and its constructiveness has further applications, too.

Lemma 4.6. *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric, positively skew-supermodular function and $m \in C(p) \cap \mathbb{Z}^V$. If $M_p = \max\{p(X) : X \subseteq V\} > 1$ then there is an admissible splitting-off.*

Proof. Let Y be a minimal set satisfying $p(Y) = M_p$. By symmetry, $p(V - Y) = M_p$, too, so we can choose a minimal set $Z \subseteq V - Y$ satisfying $p(Z) = M_p$. Since $M_p \geq 1$ we can choose $y \in Y, z \in Z$ with $m(y), m(z) > 0$. We claim that the splitting at y and z is admissible. Assume that it is not and consider a dangerous set X containing y and z (note that $p(X) > 0$). Since $m(X - Y) \leq m(X) - m(y) \leq m(X) - 1$ and $p(Y - X) < M_p$ by the minimality of Y, X and Y cannot satisfy $(-)$, since that would mean $m(X) - 1 + M_p \leq p(X) + p(Y) \leq p(X - Y) + p(Y - X) < m(X - Y) + M_p \leq m(X) - 1 + M_p$, a contradiction. So X and Y must satisfy $(\cap\cup)$, which implies (using $m(X \cap Y) = m(X) - m(X - Y) \leq m(X) - m(z) \leq m(X) - 1$) that $p(X \cup Y) = M_p$ and $m(X - Y) = 1$, since $m(X) - 1 + M_p \leq p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y) \leq m(X \cap Y) + M_p \leq m(X) - 1 + M_p$. Now $X \cup Y$ and Z cannot satisfy $(-)$ since this would give $p(Z - (X \cup Y)) = M_p$, contradicting the minimality of Z. Therefore $X \cup Y$ and Z satisfy $(\cap\cup)$ implying that $p(Z \cap (X \cup Y)) = M_p$, which is only possible if $Z \subseteq X \cup Y$. But $2 \leq M_p = p(Z) \leq m(Z) \leq m(X - Y) = 1$ gives a contradiction. □

Let us mention an important consequence of this lemma. If there is no admissible splitting-off, then $p \leq 1$ and every pair $u, v \in V^+$ is in a dangerous set X: this means that $p(X) = 1$ and $m(X) = 2$, hence $m \leq 1$. The following corollary is immediate: it generalizes a Theorem 3.2 of Szigeti and in a special case it was also observed by Ben Cosh (see [15]).

Corollary 4.7. *If p is a symmetric, positively skew-supermodular function and $m \in C(p) \cap \mathbb{Z}^V$ then there is a hypergraph H covering p satisfying the degree-specification m that contains at most one hyperedge of size greater than 2.*

Proof. If there is an admissible splitting-off then the statement follows by induction on $m(V)$. Otherwise $p \leq 1$ by Lemma 4.6: observe that a hypergraph containing the only hyperedge V^+ covers p in this case by the symmetry of p. □

Consider the following greedy algorithm for the degree-specified covering problem (note that this algorithm can be implemented to run in polynomial time **only** with a maximizing oracle for p).

Algorithm GREEDYCOVER

begin

INPUT A **symmetric, positively skew-supermodular** function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ (given with a maximizing oracle) and $m \in C(p) \cap \mathbb{Z}^V$.

OUTPUT A graph $G = (V, E)$ and a hyperedge e such that the hypergraph $G + e$ covers p and $d^+_{G+e}(v) = m(v)$ for every $v \in V$.

1.1. Initialize $G = (V, \emptyset)$.
1.2. While there exists an admissible pair u, v do
1.3. Let $m = m - \chi(u) - \chi(v)$ and $p = p - d_{(V,\{(u,v)\})}$ and $G = G + (uv)$.
1.4. Output G and e where $\chi_e = m$.
end

Clearly, if one can test membership in $C(p - d_G)$ in polynomial time for any graph G (which can be done using a maximizing oracle) then this algorithm can be implemented to run in polynomial time using the following observations. If a pair $u, v \in V^+$ is admissible at some point, then by a binary search we can determine the maximum number of admissible splittings at u and v. Furthermore, if a pair of positive nodes u, v is not admissible at some point, then it will not become admissible later, since dangerous sets remain dangerous.

We say that the algorithm **got stuck** if the hyperedge e in the output is of size greater than 0. Using the symmetry of p we can show that the algorithm cannot get stuck with $m(V) = 2$.

4.2.1 General observations on the stuck situation

We can read out many things about the situation when the algorithm GREEDYCOVER gets stuck from Lemma 4.6. In the following lines we will do this. So for the rest of Section 4.2 let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be symmetric and positively skew-supermodular, $m \in C(p)$ and we assume that there is no further admissible splitting off. By Lemma 4.6 and the remarks following it, every pair $u, v \in V^+$ is in a dangerous set X: since $p \leq 1$ this means that $p(X) = 1$ and $m(X) = 2$, hence $m \leq 1$. The interesting case for us will be the case when the splitting procedure gets stuck with $m(V) \geq 4$: we either want to find a graph, when $m(V)$ has to be even, or we want to solve a rank-respecting problem when getting stuck with $m(V) = 3$ is satisfactory. In the rest of Section 4.2 we assume that we are at this stuck situation with $m(V) \geq 4$.

Observe that the algorithm GREEDYCOVER can be modified in an obvious way if $G + e$ is not a feasible output (for example e is too big): we can replace e with any connected hypergraph on V^+. For example if we are only allowed to use graph edges then we can notice that with $m(V) - 1$ graph edges we can finish the procedure: any spanning tree on V^+ will cover p. However, we could possibly cover p with less edges, as the example $p(X) = 1$ if $|X| \in \{1, 2, |V| - 2, |V| - 1\}$ (and $p(X) = 0$ otherwise) shows. Though there is a lower bound: one needs at least $\lceil 2(m(V) - 1)/3 \rceil$ edges to finish the procedure. The special case of the following lemma when m is minimal was also proved by Nutov in [36], who used it to devise approximation algorithms for the problem of covering a symmetric, positively skew-supermodular function with a minimum number of graph edges.

Lemma 4.8. *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric, positively skew-supermodular function and $m \in C(p) \cap \mathbb{Z}^V$. Assume that there is no admissible splitting-off. Then any (inclusionwise) minimal graph K covering p has at least $\lceil 2(m(V)-1)/3 \rceil$ edges.*

Proof. We claim that we can assume that the edges of F connect positive nodes: consider any edge $e = (xy) \in E(K)$, where at least one of x and y is not positive. Since $K - e$ does not cover p, the function $p' = p - d_{K-e}$ has positive values, however obviously $p' \le p \le 1$. We claim that the family $\mathcal{F} = \{X \subseteq V : x \in X, y \notin X, p'(X) = 1\}$ is closed under intersection and union. Let $X, Y \in \mathcal{F}$: then p' cannot satisfy $(-)$ for X and Y, since then K would not cover p. So $(\cap\cup)$ holds for X and Y and function p', which implies that $X \cap Y, X \cup Y \in \mathcal{F}$, as claimed. Since $m \in C(p) \cap \mathbb{Z}^V$, there must be a positive node $x_0 \in \cap\mathcal{F}$ and a positive node $y_0 \in V - \cup\mathcal{F}$, so $K' = K - e + (x_0y_0)$ also covers p and iterating this we arrive at a graph (with the same number of edges) that has only edges between positive nodes.

It is clear that $K[V^+]$ cannot contain a component of cardinality 2, since any pair of positive nodes is contained in a dangerous set, therefore an edge of K must leave this set. If m is not minimal then it is possible that there exists a $v \in V^+$ that is not contained in a tight set, but in this case any other $u \in V^+$ must be contained in a tight set, since if $u_1, u_2 \in V^+$ then $X_{u_1,v}$ and $X_{u_2,v}$ cannot satisfy $(\cap\cup)$ (since their intersection is not tight), so they satisfy $(-)$, implying that u_1 and u_2 are both contained in a tight set, as claimed. Therefore every component of $K[V^+]$ must be of cardinality at least 3, except for at most one singleton component (however, if m is minimal, then there is no such singleton component). So if $\mathcal{C}$ denotes the set of these components then $|\mathcal{C}| \le (m(V)-1)/3 + 1$. Using this we have

$$|E(K)| \ge \sum_{C \in \mathcal{C}} (|V(C)| - 1) \ge m(V) - (m(V)+2)/3 = 2(m(V)-1)/3. \qquad \square$$

Let us give a lemma that will be useful later. Assume $x_0, x_1, x_2 \in V$ are three different positive nodes and X_0, X_1, X_2 are three dangerous sets blocking them with $x_i \in X_j \cap X_k$ for any $\{i, j, k\} = \{0, 1, 2\}$. (Since we assume that $m(V) \ge 4$ the three sets X_0, X_1, X_2 are pairwise crossing here.) We will say that X_0, X_1 and X_2 form a **blocking-triangle**. We say that X_0 is **slim** if $X_0 \cap X_1 \cap X_2 = \emptyset$ and $X_0 - (X_1 \cup X_2) = \emptyset$.

Lemma 4.9 (Slimming Lemma)**.** *Assume that X_1 and X_2 satisfy* $(\cap\cup)$*. Then $(X_0 - (X_1 \cap X_2)) \cap (X_1 \cup X_2)$ is also dangerous and blocks x_1, x_2.*

Proof. Since X_1 and X_2 satisfy $(\cap\cup)$, $p(X_1 \cap X_2) = p(X_1 \cup X_2) = 1$. Now $X_1 \cap X_2$ and X_0 cannot satisfy $(\cap\cup)$, since that would imply that $p(X_0 \cap X_1 \cap X_2) = 1$, but

$m(X_0 \cap X_1 \cap X_2) = 0$. This implies that $p(X_0') = 1$ where $X_0' = X_0 - (X_1 \cap X_2)$. Now X_0' and $X_1 \cup X_2$ cannot satisfy $(-)$, since that would give $p(X_0' - (X_1 \cup X_2)) = 1$ contradicting $m(X_0' - (X_1 \cup X_2)) = 0$. So we obtain from $(\cap\cup)$ that $p(X_0' \cap (X_1 \cup X_2)) = 1$ and clearly $x_1, x_2 \in X_0' \cap (X_1 \cup X_2)$. □

We note that the family of sets blocking a fixed pair of nodes $u, v \in V^+$ is closed under union and intersection. Let us denote the unique minimal member of this family by X_{uv}. Observe, that for 4 different nodes $u, v, x, y \in V^+$ we have $X_{uv} \cap X_{xy} = \emptyset$: they cannot satisfy $(\cap\cup)$ since $m(X_{uv} \cap X_{xy}) = 0$, so they satisfy $(-)$, and then by minimality they must be disjoint.

4.2.2 Simple proofs

In this subsection we give simple proofs of classical results in order to demonstrate the simplicity of our approach. First we give a simple proof of a special case of a theorem of Benczúr and Frank. They proved in [5] that the problem of covering a symmetric, positively crossing supermodular set function by a minimum number of graph edges can be solved in polynomial time (if the function is given with a maximizing oracle): a complete proof of their theorem will be given in Section 5.3. In a special case this problem can be solved greedily. Many proofs below consider the situation when the Algorithm GREEDYCOVER gets stuck. In most of the cases we can assume that this is already the case in the beginning, since after some steps we are again at an instance of our starting problem: an example of this is Theorem 4.10. Note that a symmetric positively crossing supermodular function is also positively skew-supermodular (which is not necessarily the case without the symmetry). Furthermore, a symmetric, positively crossing supermodular function satisfies both $(\cap\cup)$ and $(-)$ if X and Y are crossing p-positive sets.

Theorem 4.10. *Let $p_0 : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric, positively crossing supermodular function that does not take 1 as value, and $G = (V, E)$ be an arbitrary graph. Then the Algorithm GREEDYCOVER does not get stuck with input $p = p_0 - d_G$ and arbitrary $m \in C(p) \cap \mathbb{Z}^V$ if $m(V)$ is even.*

Proof. Assume that the algorithm GREEDYCOVER gets stuck (at start). Then $m(V) \geq 4$ must hold. Consider a blocking triangle X, Y, Z. By Lemma 4.6 and the observations above, any pair of this three sets must satisfy $(\cap\cup)$ and $(-)$ for p with equality. Using the Slimming Lemma (Lemma 4.9) we can assume that X, Y and Z are all slim. However, $p(X \cap Y) = p_0(X \cap Y) - d_G(X \cap Y) = 1$ and $p_0(X \cap Y) \neq 1$ implies that there must be an edge in G leaving $X \cap Y$. But in presence of such an edge we are able to find two sets out of X, Y, Z that cannot satisfy $(-)$ or $(\cap\cup)$ with equality by (1.5) and (1.6). □

The above theorem includes the classical splitting theorem of Lovász that can be used for global edge-connectivity augmentation of graphs.

Lemma 4.11 (Lovász' lemma)**.** *Let $G = (V + s, E)$ be k-edge-connected in V, where $k \geq 2$. Assume $d_G(s)$ is even. Then there exists a splitting-off at s that preserves k-edge-connectivity in V.*

Proof. Let $G' = G[V]$ and $p : 2^V \to \mathbb{Z}$ defined by $p(X) = k - d_{G'}(X)$ for any $\emptyset \neq X \neq V$ and $p(\emptyset) = p(V) = 0$. Let $m(v) = d_G(s, v)$ for any $v \in V$. With these notations the lemma follows from Theorem 4.10. $\square$

Next we give a simple proof of Mader's classical splitting lemma.

Proof of Lemma 4.2. If there is no cut edge incident to s then $\lambda_G(u, v) \geq 2$ for any pair of s-neighbours u, v. Let us define $R(X) = \max\{\lambda_G(x, y) : x \in X, y \in V - X\}$ for any X with $\emptyset \neq X \neq V$ and $R(\emptyset) = R(V) = 0$ and $p(X) = R(X) - d_{G[V]}(X)$ for any $X \subseteq V$. Let $m(v) = d_G(s, v)$ for any $v \in V$. By Theorem 2.10, the function R (and consequently p) is a symmetric and skew-supermodular function. By assumption, $m(X) \geq p(X)$ holds for every $X \subseteq V$. Assume that there is no splitting-off and take a blocking triangle X, Y, Z consisting of **maximal** dangerous sets. Consider the following two cases.
Case I.: Assume that two of these three sets (wlog. X and Y) satisfy $(\cap\cup)$. Then, using the Slimming Lemma, substitute Z by $Z' = (Z - (X \cap Y)) \cap (X \cup Y)$. Let $R(Z') = \lambda_G(z, v)$ with $z \in Z'$ and $v \in V - Z'$ and assume wlog. that $z \in X \cap Z'$ implying $R(Z') \leq R(X \cap Z')$. Since there is no cut edge incident to s, $d_G(Y \cap Z') \geq R(Y \cap Z') \geq 2$. Then $d_G(Z') - 1 \leq R(Z') \leq R(X \cap Z') \leq d_G(X \cap Z') = d_G(Z') - d_G(Y \cap Z') + d_G(X \cap Z', Y \cap Z') \leq d_G(Z') - 2 + d_G(X \cap Z', Y \cap Z')$ implies that $d_G(X \cap Z', Y \cap Z') > 0$, but then X and Y cannot satisfy $(\cap\cup)$ with equality by (1.5).
Case II.: Assume that X, Y and Z pairwise satisfy $(-)$. This implies that $p(X - Y) = 1$, consequently Z and $X - Y$ cannot satisfy $(-)$, since $m((X - Y) - Z) = 0$. Thus they satisfy $(\cap\cup)$ which implies by the maximality of Z that $X - (Y \cup Z) = \emptyset$. Similarly we can prove that $Y - (Z \cup X) = Z - (X \cup Y) = \emptyset$. Using that there is a neighbour of s not in $X \cup Y \cup Z$ we can deduce that $R(X \cup Y \cup Z) \geq 2$. However, since X, Y and Z pairwise satisfy $(-)$ with equality, there must not be an edge of $G[V]$ leaving $X \cup Y \cup Z$ by (1.6). But this would imply that $p(X \cup Y \cup Z) \geq 2$, contradicting Lemma 4.6. $\square$

Finally we will give a simple proof of a theorem of Bang-Jensen, Frank and Jackson [2] on undirected splitting-off in mixed graphs: the $k = l$ case is a special case of Theorem 3.2 of [2], so we also manage to extend slightly this special case. We mention that the authors of [2] proved a more general theorem on the augmentation of mixed graphs that also contained Mader's Lemma as a special case, but we only concentrate on a special case of their theorem, which is identical to the $k = l$ case of our theorem below. Interestingly, if either $k = 1$ or $l = 1$ in this problem then the answer is different and this is not covered by the following theorem.

Theorem 4.12 (Bang-Jensen, Frank, Jackson)**.** *Let $M = (V + s, E)$ be a mixed graph and assume that s is only incident with undirected edges. Let $r \in V$ and $k, l \in \mathbb{N} - \{1\}$ integers and assume that $\lambda_M(r, v) \geq k$ and $\lambda_M(v, r) \geq l$ for any $v \in V$. Then there exists a splitting-off at s preserving this property, provided that $d_M(s) > 3$.*

Proof. We can assume that $M - s$ is a digraph (by substituting undirected edges by oppositely directed pairs of arcs): let us denote this digraph by $D = (V, A)$ and let $m(v) = d_M(s, v)$ for any $v \in V$. Let the function q be defined by $q(\emptyset) = q(V) = 0$, $q(X) = k - \varrho_D(X)$ for any nonempty $X \subseteq V - r$ and $q(X) = l - \varrho_D(X)$ for any $X \subsetneq V$ with $r \in X$ (i.e. $q = q_{D,r,k,l}$ as defined in (2.2)). Then one can check that q is crossing supermodular and thus $p = q^s$ is skew-supermodular. Since M is (k, l)-arc-connected from r (apart from s), $m(X) \geq p(X)$ for any $X \subseteq V$. Assume that there is no splitting-off. Consider a blocking triangle X, Y, Z. We can assume without loss of generality that either $q(X) = q(Y) = 1$ or $\overline{q}(X) = \overline{q}(Y) = 1$ so X and Y must satisfy $(\cap\cup)$ with equality, implying that $d_D(X, Y) = 0$. By Lemma 4.9 we can assume that Z is slim. If $r \notin Z$ then either $\varrho_D(Z) = k - 1$ or $\delta_D(Z) = l - 1$: assume the former, the other case being analogous. But $\varrho_D(Z \cap X) \geq k - 1$ and $\varrho_D(Z \cap Y) \geq k - 1$ together with $k \geq 2$ implies that $d_D(X, Y) > 0$, a contradiction. If $r \in Z$ (wlog. $r \in Z \cap X$) then either $\varrho_D(Z) = l - 1$ or $\delta_D(Z) = k - 1$: assume the former, and observe that $\varrho_D(Z \cap X) \geq l - 1$ and $\varrho_D(Z \cap Y) \geq k - 1 > 0$ again imply that $d_D(X, Y) > 0$, thus yield a contradiction. □

4.3 Stuck situation for special skew-supermodular functions

In this section we want to characterize the stuck situation if the symmetric, positively skew-supermodular function p is of form q^s with some special function q. Throughout in Section 4.3 we will assume (4.1) by Lemma 2.17 on contraction:

$$p\text{-positive tight sets are singletons.} \tag{4.1}$$

Recall that for a pair $u, v \in V^+$ the unique minimal set blocking them is denoted by X_{uv}. Observe that for four nodes $x, y, u, v \in V^+$

$$X_{xy}(-)\, X_{yu} \text{ and } X_{yu}(-)\, X_{uv} \Rightarrow |X_{xy}| = |X_{yu}| = |X_{uv}| = 2. \tag{4.2}$$

This is true since p-positive tight sets are singletons and $X_{xy} \cap X_{uv} = \emptyset$ as we noted in the end of Section 4.2.1. For the subsequent two subsections let us introduce some notations. If p is the symmetrized of a function q then for any set X either $p(X) = q(X)$ or $p(X) = q(\overline{X})$ (possibly both). In the former case we say that X is of q**-type**, in the latter we say that X is of $\overline{q}$**-type** (so X can be of both types). We introduce two (undirected, simple) graphs on the set of positive nodes: the edge set of the q**-graph** ($\overline{q}$**-graph**) consists of the pairs u, v of positive nodes having $q(X_{uv}) = 1$ ($\overline{q}(X_{uv}) = 1$, respectively). Since there is no admissible splitting, the union of these two graphs is the complete graph (on the set of positive nodes), and an edge may belong to both graphs. We will call this 2-edge-coloured complete graph **the** $q\overline{q}$**-graph**.

4.3.1 Crossing supermodular functions

In this subsection we characterize the stuck situation if p is the symmetrized of a positively crossing supermodular function q. Recall that a set function $q : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is called *positively crossing supermodular* if it satisfies $(\cap\cup)$ whenever X and Y are crossing and $q(X), q(Y)$ are both positive. One can check that the complement of a positively crossing supermodular function is also positively crossing supermodular, and the symmetrized of a positively crossing supermodular function is positively skew-supermodular.

If two p-positive crossing sets X and Y are of the same type then they will satisfy $(\cap\cup)$. If furthermore $p(X) = p(Y) = 1$ then their intersection and union is also of the same type as X and Y and $p(X \cup Y) = p(X \cap Y) = 1$ (here we use that $p \leq 1$). On the other hand if X and Y are of different types then $p(X - Y) = p(Y - X) = 1$. Also note that from any three sets there are two of the same type.

If p is symmetric and positively crossing supermodular, then it is easy to check that every node is positive by (4.1) (one can find examples showing that this does not hold in general, if only the positively skew-supermodularity of p is assumed). However we will prove this statement for our more general case, when p is the symmetrized of a positively crossing supermodular function q. First it is useful to prove the following lemma.

Lemma 4.13. *If p is the symmetrized of a positively crossing supermodular function q then $|X_{uv}| = 2$ for any $u, v \in V^+$.*

Proof. Assume that there are nodes $x, z \in V^+$ such that $|X_{xz}| > 2$. By possibly complementing q we can assume that X_{xz} is of q-type. Let $y \in V - X_{xz}$ be another positive node. We claim that X_{xy} must be of q-type, too. If not, then $X_{xz} - X_{xy} = z$, $X_{xy} - X_{xz} = y$, since they are tight. But then X_{yz} cannot be of q-type (since this would imply $X_{yz} \cap X_{xz} = z$ and $X_{xy} - X_{yz} = x$, a contradiction), neither of $\overline{q}$-type (for a similar reason). So we have proved that for any $y \in V^+ - \{x, z\}$ the set X_{xy} is of q-type. So the union of these sets $Y = \bigcup_{y \in V^+ - \{x,z\}} X_{xy}$ is also of q-type, and has $p(Y) = q(Y) = 1$. However this implies that $1 = p(V - Y) = m(z) = m(V - Y)$, so it is tight, which contradicts $|X_{xz}| > 2$ (note that $Y \cap X_{xz} = x$). □

The lemma implies that the edge set of the q-**graph** ($\overline{q}$-**graph**) consists of the pairs u, v of positive nodes having $q(\{u, v\}) = 1$ ($\overline{q}(\{u, v\}) = 1$, respectively). Observe that a non-singleton connected component $X \neq V$ of the q-graph is also of q-type and has $q(X) = 1$ by Claim 1.9 (and similarly for the $\overline{q}$-graph). This immediately implies the result promised before.

Lemma 4.14. *If p is the symmetrized of a positively crossing supermodular function q then every node is positive.*

Proof. Suppose not, then the set of positive nodes $V^+ \neq V$ must be connected in at least one of the two graphs (since the union of two disconnected graphs cannot be the complete graph), so $p(V^+) = 1$. But then $p(V - V^+) = 1$ by the symmetry, contradicting $m(V - V^+) = 0$. □

What is more, this implies the following surprising observation.

Lemma 4.15. *If p is the symmetrized of a supermodular function q, then $p(X) = 1$ for any X with $\emptyset \neq X \neq V$ (i.e. $q(X) = 1$ or $q(V - X) = 1$ for every such set).*

Proof. By the preceding argument, any non-singleton $X \subsetneq V$ must be connected in at least one of the two graphs, so has $p(X) = 1$ (it is also easy to see for singletons, using $m(V) \geq 4$). □

Consequently we have a crossing family $\mathcal{F}$ containing all sets with q value 1, and the family of the complements of this family $\mathrm{co}(\mathcal{F})$ (these are the sets with $\overline{q}$ value 1), and the union of these two families is $2^V - \{\emptyset, V\}$. In the following theorem we will characterize such families (for sake of brevity we will also add $\emptyset$ and V to the family: these sets can always be added to or removed from a crossing family).

Let $x \in V$ and let $X_1, \ldots, X_t$ be $t \geq 1$ pairwise disjoint subsets of $V - x$ (possibly $t = 1$ and $X_1 = \emptyset$). We introduce the following family:

$$\mathcal{F}_{x,X_1,\ldots,X_t} = \{X \subseteq V : x \in X \text{ or } X \subseteq X_i \text{ for some } i \in 1, \ldots, t\}.$$

Theorem 4.16. *Let $\mathcal{F} \subseteq 2^V$ be a crossing family with $\emptyset, V \in \mathcal{F}$ that satisfies $\mathcal{F} \cup \mathrm{co}(\mathcal{F}) = 2^V$. Then either V has exactly four elements and $\mathcal{F} = 2^V \setminus \{\{y, z\}\}$ for some $y \neq z$, $y, z \in V$ or there exists a node $x \in V$ and $X_1, \ldots, X_t$ pairwise disjoint subsets of $V - x$ for some $t \geq 1$ such that either $\mathcal{F}$ or $\mathrm{co}(\mathcal{F})$ is equal to $\mathcal{F}_{x,X_1,\ldots X_t}$ or $\mathcal{F}_{x,X_1,\ldots X_t} \cup \{V - x\}$.*

Proof. We can clearly assume that V has at least 3 elements. We introduce 2 (simple undirected) graphs on V: for sake of simplicity we call them **blue** and **red**. The blue graph is $B = (V, \{(u,v) : \{u,v\} \in \mathcal{F}\})$, and the red is $R = (V, \{(u,v) : \{u,v\} \in \mathrm{co}(\mathcal{F})\})$ (so some edges might belong to both graphs). It might seem that these graphs don't have every information on $\mathcal{F}$, but it turns out that they almost do. Again, we have that the union of these two graphs is the complete graph, and a non-singleton connected component $X \neq V$ of B is in $\mathcal{F}$ (so $V - X$ is in $\mathrm{co}(\mathcal{F})$). This implies that if $B[V - \{u,v\}]$ is connected for nodes $u \neq v$, then $(u,v) \in R$, and vice versa. If $(u,v) \in B$ then we will say that this edge is **blue**, if $(u,v) \notin R$ then we will say that this edge is **pure blue**.

Claim 4.17. *There is a node $x \in V$ such that either B or R contains the edges (x,v) for every $v \in V - x$.*

Proof. Assume indirectly that every node $v \in V$ is entered by a pure red edge and by a pure blue edge, too. Consider an edge (u,v) that is pure blue: this means that $B[V - \{u,v\}]$ is disconnected, so there is a bipartition X, Y of $V - \{u,v\}$ such that every edge is pure red between X and Y. Assume wlog. that the pure red neighbour x of v is in X and consider two cases.

CASE I. $|X| \geq 2$. Since $R[V - \{v,x\}]$ must be disconnected, every edge of the form (u,y) must be pure blue for any $y \in V - \{u,v,x\}$. So the pure red edge entered by u must be the edge (u,x). Now consider any $x' \in X - x$: since $B[V - \{x,x'\}]$ is connected, this edge is red, but then x is not entered by a pure blue edge, a contradiction.

CASE II. $X = \{x\}$. Then there is a bipartition Y_1, Y_2 of $Y + u$ such that every edge between Y_1 and Y_2 is pure blue. Assume that $u \in Y_1$ and consider any $y \in Y_1 - u$: since $R[V - \{u,y\}]$ is connected, this edge is blue. Then the only possibility for a pure red edge incident to u is necessarily the edge (u,x) which again means that there is no pure blue edge leaving x, finishing the proof of the claim. ■

So consider the vertex x given by this claim and assume wlog. that (x,v) is blue for any $v \in V - x$. We distinguish again two cases.

CASE I. There exist two properly intersecting sets $Y, Z \in \mathcal{F}$ such that $Y \cup Z = V - x$. We claim that this Y and Z can be chosen such that their symmetric difference is of cardinality two. Indeed, for any $y \in Y - Z$ the set $V - \{x,y\}$ also belongs to $\mathcal{F}$ since Z and $Y - y = (Y - y + x) \cap Y$ both belong to $\mathcal{F}$, they are crossing and this is their union. So Z can be substituted by $Z' = V - \{x,y\}$ and similarly Y can be substituted by $Y' = V - \{x,z\}$ for some $z \in Z - Y$. Now if $|V| > 4$ then this implies that $\mathcal{F} = 2^V$. To prove this consider an arbitrary $A \subseteq V - x$: if $A \subseteq Y'$ then clearly $A \in \mathcal{F}$, since A is the intersection of the crossing sets $A + x \in \mathcal{F}$ and $Y' \in \mathcal{F}$. Similarly, $A \subseteq Z'$ implies that $A \in \mathcal{F}$. Therefore we assume that $y, z \in A$. If $A \neq \{y, z\}$ then $A \in \mathcal{F}$ follows since A is the union of $A - y$ and $A - z$ that are crossing sets in $\mathcal{F}$. The only remaining case is $A = \{y, z\}$. Let $v \in V - \{x, y, z\}$ arbitrary, then A is the intersection of $\{v, y, z\} \in \mathcal{F}$ and $\{x, y, z\} \in \mathcal{F}$, that are crossing sets if $|V| > 4$, implying that $A \in \mathcal{F}$, as claimed. If $|V| = 4$ then $\mathcal{F}$ can also be $2^V \setminus \{\{y, z\}\}$.

CASE II. There does not exist two properly intersecting sets $Y, Z \in \mathcal{F}$ such that $Y \cup Z = V - x$. Let the maximal sets of $\mathcal{F}$ properly contained in $V - x$ be $X_1, X_2, \ldots, X_t$: these are pairwise disjoint and since $\emptyset \in \mathcal{F}$, we have $t \geq 1$. One can simply check that $\mathcal{F}$ is either $\mathcal{F}_{x,X_1,\ldots X_t}$ or $\mathcal{F}_{x,X_1,\ldots X_t} \cup \{V - x\}$, as claimed above. □

A simple corollary that is worth mentioning is the following.

Theorem 4.18. *Let $\mathcal{F} \subseteq 2^V$ be a ring family with $\emptyset, V \in \mathcal{F}$ that satisfies $\mathcal{F} \cup \mathrm{co}(\mathcal{F}) = 2^V$. Then there exists a node $x \in V$ and a (possibly empty) set $X_1 \subseteq V - x$ such that either $\mathcal{F}$ or $\mathrm{co}(\mathcal{F})$ is equal to $\mathcal{F}_{x,X_1}$.*

4.3.2 Crossing negamodular functions

Recall that a set function $q : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is called *positively crossing negamodular* if it satisfies $(-)$ whenever X and Y are q-positive crossing subsets. Note that the symmetrized of a positively crossing negamodular function is positively skew-supermodular, but the complement of a positively crossing negamodular function is not necessarily positively crossing negamodular. An important special case is a **monotone decreasing** function: by that we mean a function q that satisfies $q(\emptyset) \leq 0$ but $q(X) \geq q(Y)$ for any $\emptyset \subsetneq X \subseteq Y \subseteq V$.

In this section we want to characterize the stuck situation if $p = q^s$ with a positively crossing negamodular function q. Recall that for a pair $u, v \in V^+$ the unique minimal set blocking them is denoted by X_{uv}. If X_{xy}, X_{yu} and X_{uv} are of the same type for four different positive nodes x, y, u, v then they all must be of cardinality two by (4.2).

As the NP-completeness of the general node-to-area connectivity augmentation problem in graphs shows, we cannot expect a nice characterization of the stuck situation of the Algorithm GREEDYCOVER in general, if $p = q^s$ with an arbitrary crossing negamodular function q. Therefore we restrict ourselves to some special positively crossing negamodular functions in this section: we assume that

$$\begin{gathered} q = R - d_{H_0} \text{ with a positively crossing negamodular function } R \\ \text{satisfying } R(X) \neq 1 \text{ for any } X \subseteq V \text{ and an arbitrary hypergraph } H_0. \end{gathered} \tag{4.3}$$

The following lemma characterizes the stuck situation of the Algorithm GREEDYCOVER with the input $p = q^s$ and $m \in C(p) \cap \mathbb{Z}^V$. A hyperedge of H_0 is called an *m-large hyperedge* if it contains at least two nodes of V^+.

Lemma 4.19. *If $p = q^s$ for a function q given with (4.3), $m \in C(p) \cap \mathbb{Z}^V$, there is no admissible splitting-off and $m(V) \geq 5$ then there exists an m-large hyperedge. Furthermore, the number of positive nodes that are avoided by an m-large hyperedge is at most one.*

Proof. Assume that there is no m-large hyperedge. Apply $(-)$ for X_{ab} and X_{bc} and p (with an arbitrary $a, b, c \in V^+$) to get that every $a \in V^+$ is contained in a non-singleton hyperedge e. We claim that neither the q-graph nor the $\overline{q}$-graph can contain a path consisting of 3 edges. Assume indirectly that for some four nodes $x, y, u, v \in V^+$ the sets X_{xy}, X_{yu}, X_{uv} are all of the same type: then (4.2) gives that they all are of cardinality 2. But then X_{xy} and X_{yu} cannot satisfy $(-)$ with equality by the nonsingleton hyperedge containing y, proving our claim. One can check that the edge set of a complete graph on at least 5 nodes cannot be decomposed into 2 sets such that neither of them contains a path of 3 edges, so there must be an m-large hyperedge.

Assume that there is an m-large hyperedge e that avoids $x \in V^+$. Since e is m-large, there exist $u, v \in V^+ \cap e$. If the sets X_{xu} and X_{xv} satisfied $(\cap\cup)$ then they would have to satisfy it with equality, but their intersection would be x by (4.1) and then they cannot satisfy $(\cap\cup)$ with equality by the presence of the hyperedge e (see Lemma 1.6 (i)). Therefore X_{xu} and X_{xv} must be of the same type. If e avoids another positive node y then X_{xu} and X_{yu} cannot be of the same type for similar reasons. This implies that e cannot avoid a third positive node, so it contains at least 3 positive nodes, since $m(V) \geq 5$. Then the type of X_{uv} and X_{ux} must be different, since they cannot satisfy $(-)$ with equality because of the edge e that is not contained in X_{uv}. But then the type of X_{uv} and X_{uy} would be the same, which cannot hold for the same reason, so e cannot avoid the second positive node y. Furthermore, these observations on the $q\overline{q}$-graph show that x can be the only positive node that is avoided by an m-large hyperedge. ☐

We mention that if $m(V) = 4$ then we don't necessarily have m-large hyperedges: an example can be found in [26]. The following example shows that even if there are m-large hyperedges, they might contain 2 positive nodes if $m(V)$ is only 4. Let H_0 be the graph on node set $\{x_0, x_1, x_2, x_3, y\}$ and with edge set $\{x_0y, x_0y, x_1x_2, x_2x_3, x_3x_1\}$ (that is, a triangle and two parallel edges). The areas are $\mathcal{W} = \{\{y, x_1\}, \{y, x_2\}, \{y, x_3\}\}$ and they all have requirement 3. The only minimal degree-specification is $m = \chi_{V-y}$, there is no admissible splitting-off, and the edges of the triangle are m-large (hyper)edges.

As an application of this lemma consider the following generalization of the node-to-area connectivity augmentation problem.

Problem 4.20 (Rank-respecting node-to-area connectivity augmentation problem in hypergraphs)**.** *Given a hypergraph $H_0 = (V, \mathcal{E}_0)$ of rank at most γ, a collection of subsets $\mathcal{W}$*

of V and a function $r : \mathcal{W} \to \mathbb{Z}_+$ satisfying $r \geq 2$, find a hypergraph H of minimum total size such that $\lambda_{H_0+H}(x, W) \geq r(W)$ for any $W \in \mathcal{W}$ and $x \in V$ and the rank of H is at most γ.

If we define R_{N2A} with (2.4) and set $q = R_{N2A} - d_{H_0}$ then it is clear that $H_0 + H$ satisfies the area requirements if and only H covers q. Since R_{N2A} does not take 1 as value, we can apply Lemma 4.19 and obtain that the Algorithm GREEDYCOVER fails only slightly for this problem. Since the maximizing oracle can be implemented for this special function R_{N2A} by Lemma 2.8, the Algorithm GREEDYCOVER can be implemented to run in polynomial time for this case.

Theorem 4.21. *Let an instance of the minimum total size node-to-area connectivity augmentation problem in hypergraphs be given by the hypergraph $H_0 = (V, \mathcal{E}_0)$ of rank at most $\gamma \geq 2$, $\mathcal{W} \subseteq 2^V$ and $r : \mathcal{W} \to \mathbb{Z}_+$ with $r \geq 2$. Then the Algorithm GREEDYCOVER gives a solution that contains only graph edges and one hyperedge of size at most $\gamma + 1$, if $\gamma > 2$ and $\gamma + 2$ if $\gamma = 2$.* □

We mention that, though our proof does not rely on this, after contraction of a set T the function R_{N2A}/T can be defined with a node-to-area requirement function as follows: if R_{N2A} was defined with $\mathcal{W}$ and r then let $\mathcal{W}' = \{W \in \mathcal{W} : \ T \cap W = \emptyset\} \cup \{W - T + v_T : T \cap W \neq \emptyset\}$ and let $r'(W) = r(W)$ if $v_T \notin W$ and $r'(W') = r(W)$ if $W' = W - T + v_T$. One can check that $\mathcal{W}'$ and r' define R_{N2A}/T.

Note that for any γ there are examples where the Algorithm GREEDYCOVER would output a hyperedge of size greater than γ. For $\gamma = 2$ an example can be found in [26], for bigger values consider the following example. Let V contain $\gamma + 2$ nodes $x_0, x_1, \ldots, x_\gamma, y$ and the hypergraph H_0 contain two hyperedges $\{x_0, y\}$ and $\{x_1, \ldots, x_\gamma\}$. The areas are of the form $\mathcal{W} = \{\{y, x_i\} : \ i = 1, 2, \ldots, \gamma\}$ and $r(W) = 2$ for any $W \in \mathcal{W}$. One can check that the (only) minimal degree-specification is $m = \chi_{V-y}$ and there is no admissible splitting-off. Also note that the *SLB* cannot be achieved in this example without increasing the rank.

Figure 4.3.2 is an illustration with $\gamma = 4$. The (hyper)edges are drawn with a solid line, the areas are illustrated with dashed lines. The empty ball is the neutral node y.

If we specialize our results for graphs ($\gamma = 2$) we obtain that a greedy algorithm (the obvious modification of GREEDYCOVER) uses at most one more edge (i.e. at most two more total size) than necessary. Our results do not characterize the cases when the *SLB* in the node-to-area augmentation problem in graphs can be achieved, this can be found in [26] and [24], but they imply that a greedy algorithm can only fail by at most one (edge) for this problem.

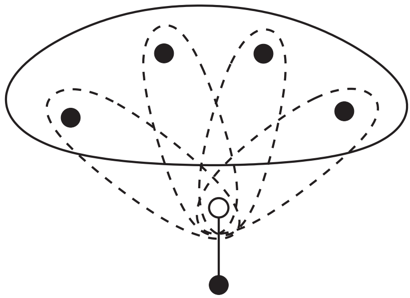

Figure 4.1: An instance of Problem 4.20 where the *SLB* cannot be achieved

A more careful analysis of the stuck situation shows that a slight modification of the Algorithm GREEDYCOVER will solve the *rank respecting node-to-area connectivity augmentation problem in hypergraphs (Problem 4.20)* optimally for $\gamma \geq 3$. In fact, we will solve the abstract version Problem 4.25 for any $\gamma \geq 3$. This can be found in Section 4.4.3.

4.4 Further applications

4.4.1 Local edge-connectivity augmentation of hypergraphs

In this section we consider the **local edge-connectivity augmentation of hypergraphs without increasing the rank**. Let $H_0 = (V, \mathcal{E}_0)$ be a hypergraph of rank at most γ, and let $r : V \times V \to \mathbb{Z}_+ \setminus \{1\}$ be a symmetric edge-connectivity requirement that does not take 1 as value. Let us define the set function R_{loc} with (2.5). Our aim is to find a hypergraph H of minimum total size such that $H_0 + H$ covers R_{loc}, that is, $\lambda_{H_0+H}(u, v) \geq r(u, v)$ for every pair of nodes u, v. Since R_{loc} is a skew supermodular function, the Algorithm GREEDYCOVER gives a solution that contains graph edges and at most one hyperedge. The question we want to answer is whether the size of this hyperedge is at most γ. One case when this is obviously not true is when $\gamma = 2$ and the *SLB* is odd: then the size of the hyperedge will be 3. The following theorem shows that this is the only exceptional case. Note that this theorem generalizes the theorem of Frank [19] on local edge-connectivity augmentation of graphs. After having proved this theorem we have been informed that Ben Cosh had also proved it in his PhD thesis [15]. We have noticed that combining our ideas with those in [15] the following simple proof can be given.

Theorem 4.22. *Let an instance of the minimum total size local edge-connectivity augmentation problem be given by a hypergraph $H_0 = (V, \mathcal{E}_0)$ of rank at most $\gamma \geq 2$, and the symmetric edge-connectivity requirement $r : V \times V \to \mathbb{Z}_+ \setminus \{1\}$. Then the hyperedge in the output of the Algorithm* GREEDYCOVER *is of size at most γ, if $\gamma > 2$, and it is of size at most 3, if $\gamma = 2$.*

Proof. We will prove that the hyperedge in the output of the Algorithm GREEDYCOVER is of size at most γ for any **minimal** input $m \in C(p) \cap \mathbb{Z}^V$: observe that this contains more general augmentation problems, e.g. the minimum node-cost version, too. We can assume that the Algorithm GREEDYCOVER is stuck already at the beginning and assume indirectly that $m(V) > \max\{3, \gamma\}$. This proof is easier told if we think of the classical description of splitting-off: instead of the degree specification m, add a new node s to the hypergraph and introduce $m(v)$ parallel edges between s and any $v \in V$ (of course, from the results above $m(v) \leq 1$ for any v, so there are no parallel edges incident to s: the positive nodes become the neighbours of s) and denote the hypergraph obtained this way with H' (i.e. H' has node set $V + s$ and it contains the hyperedges of H_0 and only graph edges incident to s). By the assumptions, H' is r-edge-connected in V. We can further assume that $r(u, v) = \lambda_{H'}(u, v)$ for any $u, v \in V$, since if we increase $r(u, v)$ then we do not create new admissible splittings (note, that possibly new sets become tight and $r(u, v) > 1$ if u and v are neighbours of s). So we assume that R_{loc} is defined by (2.5) with $\lambda_{H'}$ substituted in the place of r and $p = R_{loc} - d_{H_0}$ as before. We further assume that tight sets are singletons, which implies that $\lambda_{H'}(u, v) = \min(d_{H'}(u), d_{H'}(v))$ for any $u, v \in V$. Let t be a neighbour of s such that $d_{H'}(t) = \min\{d_{H'}(v) : v \text{ is a neighbour of } s\}$. Let u and v be neighbours of s (distinct from each other and from t). The following claim was already used in [20] (Claim 4.1): for completeness, we include a proof.

Claim 4.23. *X_{tu} and X_{tv} satisfy $(-)$ (with function R_{loc} and thus with p, too).*

Proof. Assume that they do not satisfy $(-)$ with R_{loc}: then they must satisfy $(\cap\cup)$. Thus their intersection is tight, so it is t. On the other hand, since $d_{H'}(t) \leq d_{H'}(u)$, $R_{loc}(X_{tu}) \leq R_{loc}(X_{tu} - t)$, and similarly $R_{loc}(X_{tv}) \leq R_{loc}(X_{tv} - t)$, implying that $R_{loc}(X_{tu}) + R_{loc}(X_{tv}) \leq R_{loc}(X_{tu} - t) + R_{loc}(X_{tv} - t) = R_{loc}(X_{tu} - X_{tv}) + R_{loc}(X_{tv} - X_{tu})$, so they satisfy $(-)$ after all, a contradiction. ■

So there exists a set $X \subseteq V$ containing t, such that $X_{tu} = X + u$ for any neighbour u of s (using that tight sets are singletons). Since $p(X) \leq 1$ (by Lemma 4.6) and $R_{loc}(X) \geq 2$ (using that $t \in X$), there must be a hyperedge e in H_0 entering X. We claim that this hyperedge must contain every neighbour of s (except possibly t), contradicting the hypothesis that the rank of H_0 is at most γ. Assume that it excludes a neighbour u of

s and fix an arbitrary other neighbour v of s (distinct from u and t). Since X_{tu} and X_{tv} must satisfy $(-)$ with equality for p, this implies that $e \subseteq X_{tv}$. But then X_{tu} and X_{tw} will not satisfy $(-)$ with equality for a fourth neighbour w of s. □

We mention that the minimality of m is crucial in the proof above: if m is not minimal then a simple example shows that the greedy algorithm can fail and produce a hyperedge of size $\gamma+1$. As an example, consider the following hypergraph $H_0 = (V, \{V - x\})$, where V is an arbitrary finite set and $x \in V$ is a fixed node (i.e. H_0 has only one hyperedge $V - x$). Let $r(u, v) = 2$ between any $u, v \in V - x$ and $r(x, v) = 0$ between x and an arbitrary $v \in V - x$. If we set $m(a) = 1$ for every $a \in V$, which is an admissible but not minimal degree-specification, then there does not exist an admissible splitting-off, as one can check. Thus the Algorithm GREEDYCOVER outputs the hyperedge V that has rank one greater than that of H_0.

4.4.2 Global arc-connectivity augmentation of mixed hypergraphs

In this section we give an application of the results of Section 4.3.1 about global edge-connectivity augmentation of mixed hypergraphs. If $M = (V, \mathcal{A})$ is a mixed hypergraph and $X \subseteq V$ then contracting X yields the mixed hypergraph $M/X = (V/X, \mathcal{A}/X)$ the following way: for every $a = (T_a, H_a) \in \mathcal{A}$ let $T'_a = T_a$ if $T_a \cap X = \emptyset$ and let $T'_a = T_a - X + v_X$ otherwise, similarly let $H'_a = H_a$ if $H_a \cap X = \emptyset$ and let $H'_a = H_a - X + v_X$ otherwise. Then $\mathcal{A}/X = \{a' = (T'_a, H'_a) : \ a \in \mathcal{A}\}$. Observe that $\varrho_M/X = \varrho_{M/X}$. If the root node r is in X then the contracted node v_X will become the new root node. This shows that contracting a set defines a contracted problem the natural way.

Let $M = (V, \mathcal{A})$ be a mixed hypergraph and let k, l be nonnegative integers such that $k \neq 1$ and $l \neq 1$. We assume that M is of rank at most γ. We want to make M (k, l)-arc-connected by adding an undirected, degree specified hypergraph that also has rank at most γ. Is it true that the Algorithm GREEDYCOVER will output such a hypergraph? The answer is “almost yes”: sometimes this can only be done by adding a hyperedge of cardinality $\gamma + 1$ (even for $k = l = 2$). As an example, consider the following mixed hypergraph $M = (V, \mathcal{A})$: let $|V| \geq 3$ and $x, y \in V$ be two nodes. There are 3 hyperarcs in $\mathcal{A}$: one is a digraph arc (x, y), the second is $(y, V - x - y)$ and the third is $(V - x - y, x)$. Finally let $k = l = 2$ and $\gamma = |V| - 1$. It is easy to see that the *SLB* is $|V|$ and the only way to achieve it is to add the hyperedge V.

Figure 4.4.2 is an illustration: the digraph arc and one of the other two hyperarcs are drawn with solid lines, the third hyperarc is drawn with dashed lines. We use the convention that the tails of a hyperarc are denoted by an “o” and heads by an “x” (except for the digraph arc, which is denoted by an arrow).

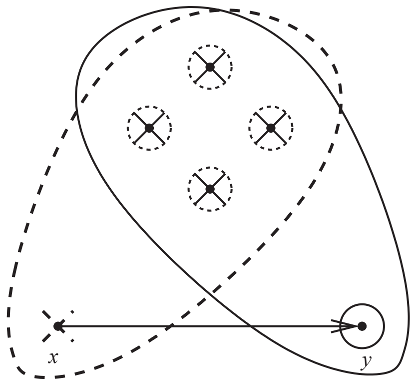

Figure 4.2: A mixed hypergraph that cannot be made 2-arc-connected with a hypergraph meeting the *SLB* without increasing the rank

However, we can prove the following result.

Theorem 4.24. *If M is of rank at most $\gamma \geq 2$ and $k, l \in \mathbb{N} - \{1\}$ are integers, then we can make M (k,l)-arc-connected greedily by the addition of graph edges and one hyperedge of size at most $\gamma + 1$.*

Proof. Let $q = q_{M,r,k,l}$ as defined in (2.2) and $p = q^s$ and let $m \in C(p) \cap \mathbb{Z}^V$. We can assume that the Algorithm GREEDYCOVER is stuck already at start. We have to prove that $m(V)$ is at most $\gamma + 1$. We can also assume that tight sets are singletons (and delete singleton hyperedges, since they are irrelevant for connectivity), so by the observations in Section 4.3.1, every node is positive. By Theorem 4.16, there is an $x \in V$ such that (by possibly reversing every hyperarc of M and switching the role of k and l) every set $X \neq V$ with $x \in X$ has $q(X) = 1$ (observe that this consequence is also true for the sporadic example on 4 nodes). First we claim that $V - x$ cannot contain hyperarcs. Assume that it does contain a hyperarc a, let v be an arbitrary head node of a, and let $X = T_a \cup H_a - v + x$ and $Y = \{v, x\}$. These sets are crossing (since $|a| < |V - x|$ by the assumption) and of q-type, but $(\cap\cup)$ cannot hold with equality for them by the presence of the hyperarc a, a contradiction. So every hyperarc of M contains x. We claim that if $v \neq x$ is a tail of a hyperarc $a = (T_a, H_a)$ satisfying $|a| \geq 3$, then $x \in H_a$ and $T_a - v - x = \emptyset$. To see this consider the crossing sets $X = a - v$ and $Y = \{v, x\}$. Then $q(X) = q(Y) = 1$ but one can check that, by the presence of the hyperarc a, $(\cap\cup)$ cannot hold with equality for X and Y, a contradiction. So the hyperarcs leaving any $v \in V - x$ all enter x and such a hyperarc cannot leave two such nodes. This implies that $\varrho(x) = \sum_{v \in V-x} \delta(v)$. If $x = r$ then $l - 1 = \varrho(x) = \sum_{v \in V-x} \delta(v) = |V - x|(l-1)$ contradicting that $|V| > 2$ and $l > 1$. On the other hand, if $x \neq r$ then $k - 1 = \varrho(x) = \sum_{v \in V-x} \delta(v) = (|V|-2)(l-1) + (k-1)$, again contradicting that $|V| > 2$ and $l > 1$. □

4.4.3 Rank-respecting augmentation of hypergraphs with negamodular constraints

In this section we will give applications of the results of Section 4.3.2: we will solve the rank-respecting node-to-area connectivity augmentation problem (Problem 4.20) for any $\gamma \geq 3$. Furthermore, we will solve a generalization of that problem, called the *rank-respecting augmentation of hypergraphs with negamodular constraints* which is defined as follows.

Problem 4.25 (Rank-respecting augmentation of a hypergraph with negamodular constraints). *Assume that $R : 2^V \to \mathbb{Z} \cup \{-\infty\}$ is positively crossing negamodular, R does not take 1 as value, and let $q = R - d_{H_0}$ with a hypergraph $H_0 = (V, \mathcal{E}_0)$ of rank at most γ. Find a hypergraph H of rank at most γ covering $q = R - d_{H_0}$ and having smallest possible total size.*

In this section we will solve this problem for any $\gamma \geq 3$ (we don't solve it for $\gamma = 2$). To this end we will slightly modify the Algorithm GREEDYCOVER and we use the observations of Section 4.3.2, where we have shown that this algorithm only fails slightly for this problem. The result of this section appeared in [7].

Assume that the Algorithm GREEDYCOVER (with input $p = (R - d_{H_0})^s$ and a minimal $m \in C(p) \cap \mathbb{Z}^V$) did not output a feasible hypergraph for Problem 4.25. Let the output of the algorithm be $G + e$ (where G is a graph, e is a hyperedge of size $\gamma + 1$). Our idea is the following: if the graph G does not contain edges at all, then it is easy to see that the *SLB* cannot be achieved (see Lemma 4.30), but one more is already enough (and it is simple to see how to reach it: any connected hypergraph on V^+ will be a good solution). Otherwise, if G contains an edge ab, then an appropriate node $c \in e$ can be deleted from e and added to ab (thus creating a hyperedge $\{a, b, c\}$ of size 3) and the hypergraph $H' = (V, E(G) - \{ab\} + \{a, b, c\} + e')$ of total size meeting the Subpartition Lower Bound is a feasible solution, where $e' = e - c$. In what follows we show that almost any choice of c will be good. In the rest of Section 4.4.3 we assume that we are at the stuck situation of the Algorithm GREEDYCOVER (so the notations p, m, V^+ are meant for this case). We will denote the output of the algorithm by $G + e$ (where G is a graph and e is the hyperedge with $\chi_e = m$) and V^+ will just be another synonym for e. The following lemma tells us the condition that we have to satisfy when choosing node c.

Lemma 4.26. *Let $p = p_0 - d_G$ with a symmetric, positively skew-supermodular function p_0 and a graph G, and let $m \in C(p)$. Assume that there does not exist an admissible splitting-off. Let $ab \in E(G)$ and $c \in V^+$ be arbitrary. Let the hypergraph G' be obtained from G by deleting the edge ab and adding the 3-hyperedge $\{a, b, c\}$, let $p' = p_0 - d_{G'}$ and let $m' = m - \chi_{\{c\}}$. Then $m' \in C(p')$ if and only if there is no set $X \subseteq V$ satisfying $p(X) = m(X) = 1$, $c \in X$ and $\{a, b\} \cap X \neq \emptyset$.*

Proof. If $m' \notin C(p')$ then there must be a set $X \subseteq V$ such that $m'(X) < p'(X)$. Since $p' \leq p \leq 1$, we get that $p'(X) = p(X) = 1$. This together with $m \in C(p)$ gives that $m(X) \geq p(X) = p'(X) = 1 > m'(X)$, implying that $c \in X$ and $\{a, b\} \cap X \neq \emptyset$, as claimed. □

The following corollary of Lemma 4.19 can be read out from its proof.

Corollary 4.27. *If there is no admissible splitting-off, tight sets are singletons and $m(V) > \gamma \geq 4$ then there exists an m-large hyperedge f and a node $x \in V^+$ such that $f = V^+ - x$. In this case either the q-graph or the $\overline{q}$-graph is the complete graph on $V^+ - x$ and the other graph is the complement (i.e. a star centered at x).* □

As we have already mentioned, if $m(V) = 4$ then we don't necessarily have m-large hyperedges: an example can be found in [26]. The example in page 64 showed that even if there are m-large hyperedges, they might contain 2 positive nodes if $m(V)$ is only 4. However, the second statement of Corollary 4.27 still holds.

Lemma 4.28. *If there is no admissible splitting-off, tight sets are singletons and $m(V) = 4$ and $\gamma \leq 3$ then there exists a special node $x \in V^+$ such that either the q-graph or the $\overline{q}$-graph is the complete graph on $V^+ - x$ and the other graph is the complement (i.e. a star centered at x).*

Proof. We have to prove that neither the q-graph, nor the $\overline{q}$-graph contains a path of 3 edges. Assume that for the four nodes $v_1, v_2, v_3, v_4 \in V^+$ the sets $X_{v_1v_2}, X_{v_2v_3}$ and $X_{v_3v_4}$ are all of the same type. By (4.2), $|X_{v_1v_2}| = |X_{v_2v_3}| = |X_{v_3v_4}| = 2$. Since $p(X_{v_2v_3}) = 1$, a hyperedge h of H_0 leaves the set $X_{v_2v_3}$. Assume wlog. that h contains v_2: since $X_{v_2v_3}$ and $X_{v_1v_2}$ satisfies $(-)$ for p with equality, by Lemma 1.6.ii the hyperedge h must contain v_1 and $h \subseteq \{v_1, v_2, v_3\}$, too (note that $|h| \leq 3$). Since $X_{v_2v_3}$ and $X_{v_3v_4}$ satisfies $(-)$ with equality, by Lemma ii h cannot contain v_3 (in fact we have proved that $h = v_1v_2$). Because of the edge h, the type of $X_{v_2v_3}$ and $X_{v_1v_3}$ must be the same. Since $p(v_3) = 1$, there must be an edge g of H_0 leaving v_3: this edge cannot leave $\{v_2, v_3, v_4\}$, since $(-)$ must hold with equality for $X_{v_2v_3}$ and $X_{v_3v_4}$. In any case, either $X_{v_2v_3}$ and $X_{v_1v_3}$, or $X_{v_1v_3}$ and $X_{v_3v_4}$ will not satisfy $(-)$ with equality. $\square$

In Corollary 4.27 and Lemma 4.28 above we have shown that there exists a unique positive node x such that either the q-graph or the $\overline{q}$-graph is the complete graph on $V^+ - x$ and the other graph is the complement (i.e. a star). This translated to the situation before contraction means, that for any $u \in V^+$ there exists a tight set $X(u)$ that was contracted (so these sets are disjoint).

In the $\gamma \geq 4$ case $X(u) \cup X(v)$ is dangerous for any $u, v \in V^+ - x$. Furthermore there exists at least one m-large hyperedge $f \in H_0$ that has size γ and satisfies that $|f \cap X(u)| = 1$ for any $u \in V^+ - x$. Let $Y = V - \cup_{u \in V^+ - x} X(u)$, so $X(x) \subsetneq Y$ (it must be a proper subset, since $p(Y) \neq 1$, since no hyperedge leaves Y). In this case it is not hard to see that every set $Y \cup X(u)$ $(u \in V^+ - x)$ is of $\overline{q}$-type: this will be proved in Lemma 4.30. It is clear, that the graph G does not have edges between the partition classes $\{X(u) :\ u \in V^+ - x\} \cup \{Y\}$.

On the other hand, **in the $\gamma \leq 3$ case** there exists a set $Z \subseteq V - \cup_{u \in V^+} X(u)$ such that $X(u) \cup X(v) \cup Z$ is dangerous for any $u, v \in V^+ - x$ and these sets all have the same type (Z is empty in the $\gamma > 3$ case). Let $Y = V - Z - \cup_{u \in V^+ - x} X(u)$: again $X(x) \subsetneq Y$ (it must be a proper subset, since $p(Y) \neq 1$, since no hyperedge leaves Y). In this case the edges of G are either induced in a member of the partition $\{X(u) :\ u \in V^+ - x\} \cup \{Y, Z\}$, or they can even go between two classes (but only between $X(u)$ and $X(v)$ or between $X(u)$ and Z, where $u, v \in V^+ - x$). In both cases we can prove the following lemma.

Lemma 4.29. *The sets $X(u)$ $(u \in V^+ - x)$ are maximal tight sets.*

Proof. Assume that there is $u \in V^+ - x$ and a tight set $X \supsetneq X(u)$. Let $v, w \in V^+ - \{u, x\}$ be arbitrary and observe that type of $X(v) \cup X(w) \cup Z$ and that of $Y \cup X(v) \cup X(w) \cup Z$ is different: this follows from (1.7) applied to $X(v) \cup X(w) \cup Z$ and $Y \cup X(v)$. This implies that $X \cap X(v) = X \cap Z = \emptyset$, since $(\cap\cup)$ for X and either of $X(v) \cup X(w) \cup Z$ and $Y \cup X(v) \cup X(w) \cup Z$ (and p) would give a contradiction. We only need to prove that $X \cap Y$ is empty. Assume that it is not and distinguish the following two cases. Clearly, the type of X and $Y \cup X(z)$ has to be the same for any $z \in V^+ - x$, otherwise $p(X \cap (X(z) \cup Y)) = 1$ would follow from $(\cap\cup)$ for X and $X(z) \cup Y$, and it would give a contradiction.

CASE I: Every set $Y \cup X(z)$ $(z \in V^+ - x)$ is of q-type. By (1.7) applied to $X(v) \cup X(w) \cup Z$ and $Y \cup X(v)$, the set $Y \cup X(v) \cup X(w) \cup Z$ would be of $\overline{q}$-type, implying that $q(X \cap Y) = 1$, a contradiction.

CASE II: Every set $Y \cup X(z)$ $(z \in V^+ - x)$ is of $\overline{q}$-type. Apply Claim 1.10 for $X_0 = X$ and the sets $X(u) \cup X(z) \cup Z$ $(z \in V^+ - x)$ to get that $p(X \cup Z \cup \bigcup_{z \in V^+ - x} X(z)) = 1$. Let $Y' = V - (X \cup Z \cup \bigcup_{z \in V^+ - x} X(z)) = Y - X$: it has p-value 1, so there must be a hyperedge h leaving it. This hyperedge cannot leave Y, so it enters $Y \cap X$. But in this case the sets $X \cup X(v) \cup Z$ and $X \cup X(w) \cup Z$ could not satisfy $(-)$ with equality, though both of them have $\overline{q}$-value 1. This contradiction finishes the proof of this lemma. $\square$

This Lemma shows that our idea can be implemented the following way: if G contains an edge ab, and $c \in V^+ - x$ is such that $X(c) \cap \{a, b\} = \emptyset$, then the hypergraph $H' = (V, E(G) - \{ab\} + \{a, b, c\} + e')$ of total size meeting the Subpartition Lower Bound is a feasible solution, where $e' = V^+ - c$. Since an arbitrary edge ab of G can intersect at most two of the sets $\{X(v) : v \in V^+ - x\}$, such a c can always be found.

Next we show that if G does not contain edges at all then the *SLB* cannot be achieved.

Lemma 4.30. *For any nonempty $U \subsetneq V^+ - x$ the set $Y \cup \bigcup_{u \in U} X(u)$ has p value 1.*

Proof. The lemma is obvious if $m(V) = 4$ by (1.7). Assume that $m(V) \geq 5$: in this case we show that every set $Y \cup X(u)$ $(u \in V^+ - x)$ is of $\overline{q}$-type. Assume not, then every such set is of q-type by Corollary 4.27, consequently $X(v) \cup X(w)$ is of $\overline{q}$-type for $v, w \in V^+ - x$. By Claim 1.10 $X(v) \cup X(w)$ is also of q-type because its complement can be built up as the union of a $\overline{q}$-type set and some q-type sets. But this also implies that $\cup_{u \in V^+ - x} X(u)$ has p-value 1, and so has its complement Y, but this cannot be the case, since no hyperedge leaves Y. Thus we have shown that every set $Y \cup X(u)$ $(u \in V^+ - x)$ is of $\overline{q}$-type and this implies the lemma by Claim 1.10. $\square$

This corollary implies that if the Algorithm GREEDYCOVER applied for Problem 4.25 outputs $G + e$ where G has no edges at all, and e is too big, then the *SLB* cannot be achieved. Let us give the pseudocode of the modified version of the Algorithm GREEDYCOVER that we have suggested.

Algorithm NEGAMODULAR_COVER
begin

INPUT A **crossing negamodular** function $R : 2^V \to \mathbb{Z}$ (given with an oracle) that satisfies $R(X) \neq 1$ for any $X \subseteq V$, and a hypergraph $H_0 = (V, \mathcal{E}_0)$ of rank at most γ (where $\gamma \geq 3$)

OUTPUT A hypergraph $H = (V, \mathcal{E})$ covering $R - d_{H_0}$ having smallest total size and rank $\leq \gamma$

1.1. Let $q = R - d_{H_0}$ and $p = q^s$ and find a minimal $m \in C(p) \cap \mathbb{Z}^V$
1.2. Initialize $H = (V, \emptyset)$

1.3. While there exists an admissible pair u, v do
1.4. Let $m = m - \chi(u) - \chi(v)$ and $p = p - d_{(V,\{uv\})}$ and $H = H + uv$
1.5. EndWhile
1.6. If $m(V) \leq \gamma$ then let $H = H + e$ where $\chi_e = m$
1.7. Else (i.e. $m(V) = \gamma + 1$)
1.8. If $E(H) = \emptyset$ then let $E(H) = \{ab, V^+ - b\}$ with arbitrary $a, b \in V^+$
1.9. Else
1.10. Let $ab \in E(H)$ be arbitrary and $c \in V^+ - x$ such that $X(c) \cap \{a, b\} = \emptyset$ (where $x \in V^+$ is the special node given by Corollary 4.27 and Lemma 4.28)
1.11. Let $E(H) = E(H) - ab + \{\{a, b, c\}, \{V^+ - c\}\}$
1.12. EndIf
1.13. EndIf
1.14. Output H and STOP
end

Chapter 5

Covering symmetric crossing supermodular functions with graph edges

A symmetric (positively) crossing supermodular function is a special case of a (positively) skew-supermodular function. Recall that set function $p\colon 2^V \to \mathbb{Z} \cup \{-\infty\}$ is called *positively crossing supermodular* if it satisfies the following inequality for every crossing pair $X, Y \subseteq V$ with $p(X), p(Y) > 0$:

$$p(X) + p(Y) \leq p(X \cap Y) + p(X \cup Y). \tag{$\cap\cup$}$$

Observe that $(\cap\cup)$ trivially holds if $X \subseteq Y$ or $Y \subseteq X$. If furthermore p is symmetric (i.e. $p(X) = p(V - X)$ for any $X \subseteq V$) then it will also satisfy the following inequality for every crossing pair $X, Y \subseteq V$ with $p(X), p(Y) > 0$:

$$p(X) + p(Y) \leq p(X - Y) + p(Y - X). \tag{$-$}$$

Again, $(-)$ will always hold if $X \cap Y = \emptyset$ or $X \cup Y = V$ (the latter by symmetry). We emphasize that we do not assume the nonnegativity of our function p.

The first half of this chapter is about **covering a symmetric (positively) crossing supermodular function with graph edges**. This problem is related to global edge-connectivity augmentation. Let us give the special cases that motivate this problem. The **global edge-connectivity augmentation problem of graphs** was defined in Section 2.1.1. It was solved by Watanabe and Nakamura in [45]. A bit more difficult problem, the **global edge-connectivity augmentation problem of hypergraphs with graph edges** was solved by Bang-Jensen and Jackson. The abstract problem of covering a symmetric (positively) crossing supermodular function with graph edges was solved by Benczúr and Frank in [5]. In Section 5.3 we give a relatively simple proof of their result. The proof contains some further simplifications due to Zoltán Szigeti.

The second part of the chapter is devoted to **covering a symmetric (positively) crossing supermodular function with partition constraints**. A special case of this problem, the **partition constrained global edge-connectivity augmentation of a graph** was solved by Bang-Jensen, Gabow, Jordán and Szigeti in [3]. This is motivated by the edge-connectivity augmentation of a bipartite graph while maintaining bipartiteness. The more general **global edge-connectivity augmentation of a hypergraph with a bipartite graph** was solved by Ben Cosh in [15]. In Section 5.4 we solve the abstract problem of covering a symmetric, positively crossing supermodular function with partition constraints and we specialize our results for the problem of **global edge-connectivity augmentation of a hypergraph with a multipartite graph**.

The structure of the chapter is the following. In Section 5.1 we briefly review the previous results on covering symmetric, positively crossing supermodular functions with graph edges. In Section 5.3 we show an algorithmic proof of Theorem 5.5 of Benczúr and Frank: this proof appeared in [8]. In Section 5.4 we solve the problem of covering symmetric, positively crossing supermodular functions with partition constraints: the results of this section appeared in [10] and [9].

5.1 Previous results - brief history

Global edge-connectivity augmentation of graphs was solved by Watanabe and Nakamura [45]. They proved the following characterization for the minimum version of the problem:

Theorem 5.1 (Watanabe, Nakamura [45]). *Let $G_0 = (V, E_0)$ be a graph, and $k \geq 2$ an integer. G_0 can be made k-edge-connected by adding at most γ new edges if and only if*

$$\sum_{Z \in \mathcal{F}} (k - d_{G_0}(Z)) \leq 2\gamma \quad \textit{for every subpartition } \mathcal{F} \textit{ of } V.$$

Note that the theorem does not hold for $k = 1$, the answer is different in this case, it depends on the number of components of G_0 (although this case is very simple, in generalizations of the above theorem this case will cause most of the difficulties). Watanabe and Nakamura showed that a minimum cardinality augmentation can be obtained in polynomial time by repeatedly increasing the edge-connectivity of the graph by one using the minimum number of edges. However, this algorithm is not strongly polynomial.

Frank [19] gave the first strongly polynomial algorithm for this problem. The algorithm relies on the following result concerning the degree specified augmentation:

Theorem 5.2. *Let $G_0 = (V, E_0)$ be a graph, $k \geq 2$ an integer, and $m : V \to \mathbb{Z}_+$ a degree specification such that $m(V)$ is even. There is a graph G such that $d_G^+(v) = m(v)$ for every $v \in V$ and $G_0 + G$ is k-edge-connected if and only if*

$$m(X) \geq k - d_{G_0}(X) \quad \textit{for every } \emptyset \neq X \subset V. \tag{5.1}$$

In [14], Cheng gave a formula on the minimum number of graph edges that can be added to an initial $(k-1)$-edge-connected hypergraph such that the resulting hypergraph is k-edge-connected. Bang-Jensen and Jackson [4] extended this result to the case when the initial hypergraph can be arbitrary. Let $c(H)$ denote the number of connected components of the hypergraph H. The min-max theorem is the following:

Theorem 5.3 (Bang-Jensen, Jackson [4]). *Let $H_0 = (V, \mathcal{E}_0)$ be a hypergraph, and k a positive integer. There is a graph G with γ edges such that $H_0 + G$ is k-edge-connected if and only if the following hold:*

$$2\gamma \geq \sum_{Z \in \mathcal{F}} (k - d_{H_0}(Z)) \quad \textit{for every sub-partition } \mathcal{F} \textit{ of } V, \tag{5.2}$$

$$\gamma \geq c(H_0 - \mathcal{E}_0') - 1 \quad \textit{for every } \mathcal{E}_0' \subseteq \mathcal{E}_0 \textit{ for which } |\mathcal{E}_0'| = k - 1. \tag{5.3}$$

Bang-Jensen and Jackson used a sophisticated splitting-off technique to prove this result, their method gives rise to a polynomial-time algorithm. Note that the $k = 1$ case is not excluded in Theorem 5.3.

The general problem of covering a symmetric (positively) crossing supermodular function with a minimum number of graph edges was solved by Benczúr and Frank. The solution of the degree specified version is the following. One natural necessary condition of the existence of a graph G satisfying the degree specification m and covering the symmetric positively crossing supermodular function p is that $m \in C(p)$ (of course we assume that $m(V)$ is even). However this is not sufficient, as the problem of making a graph connected shows. Let us introduce another necessary condition. A partition $\mathcal{X} = \{X_1, X_2, \ldots, X_t\}$ of V is called **p-full** if $p(\cup_{i \in I} X_i) > 0$ for any nonempty $I \subsetneq \{1, 2, \ldots, t\}$. The maximum cardinality of a p-full partition is the **dimension of** p and is denoted by $\dim(p)$. It is easy to see that any graph covering p must have at least $\dim(p) - 1$ edges: any graph covering p has to connect the members of any p-full partition.

Theorem 5.4 (Benczúr and Frank [5]). *Let $p_0 \colon 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric, positively crossing supermodular set function and $m_0 \in C(p_0) \cap \mathbb{Z}^V$ with $m_0(V)$ even. There exists a graph G covering p_0 with $d_G^+(v) = m_0(v)$ for any $v \in V$ if and only if $m_0(V)/2 \geq$* $\dim(p_0) - 1$.

By the methods seen so far Theorem 5.4 implies the following theorem on the minimum number of graph edges covering a symmetric, positively crossing supermodular function.

Theorem 5.5 (Benczúr and Frank [5]). *Let $p\colon 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric, positively crossing supermodular set function. The minimum number of graph edges covering p is equal to the maximum of the following two quantities:*

$$\max\{\lceil \frac{1}{2} \sum_{X \in \mathcal{X}} p(X) \rceil : \ \mathcal{X} \textit{ is a subpartition of } V\}, \tag{5.4}$$

$$\dim(p) - 1. \tag{5.5}$$

5.1.1 Partition constrained global edge-connectivity augmentation of a graph

Bang-Jensen et al. [3] were motivated by the following problem: given a bipartite graph and a positive integer k, augment this graph with a minimum number of new edges to make it k-edge-connected, while maintaining the bipartiteness. As a convenient generalization they introduced the following problem (note that the previous problem is a special case of Problem 5.6 only if the starting bipartite graph is connected, otherwise the two-colouring is not unique).

Problem 5.6 (Partition constrained global edge-connectivity augmentation of a graph). *We are given a graph $G_0 = (V, E_0)$, a partition $\mathcal{P} = \{P_1, P_2, \dots, P_r\}$ of the set V and a positive integer k. The problem is to find a graph G such that $G_0 + G$ is k-edge-connected, and G contains* ***only edges between the classes*** *of $\mathcal{P}$. In the* ***minimum version*** *the number of edges of G is to be minimized. In the* ***degree-specified version*** *G has to satisfy a given degree-specification $m \in \mathbb{Z}_+^V$.*

This problem was solved by Bang-Jensen, Gabow, Jordán and Szigeti in [3] for any $k \geq 2$ (the problem is easy for $k = 1$). In the solution of the minimum version the value $\Phi = \max\{\alpha_G, \beta_G^1, ..., \beta_G^r\}$ is a straightforward lower bounds on the minimum number of edges, where

$$\alpha_G := \max\{\lceil \tfrac{1}{2} \textstyle\sum_{X \in \mathcal{F}} k - d_G(X) \rceil : \ \mathcal{F} \text{ is a subpartition of } V\},$$
$$\beta_G^i := \max\{\textstyle\sum_{X \in \mathcal{F}} k - d_G(X) : \ \mathcal{F} \text{ is a subpartition of } P_i\} \text{ for every } i = 1, 2, \dots, r.$$

The first value α_G is related to the Subpartition Lower Bound. The value β_G^i is a lower bound since we cannot add an edge induced by P_i. However there are examples where this lower bound fails (by one). One example is when we want to make a C_4 3-edge-connected while maintaining its bipartiteness: the above bound is only 2, but we clearly need at least

3 edges. Another example is when we want to make a C_6 3-edge-connected, but we are not allowed to connect opposite nodes, then the above lower bound 3 cannot be achieved. These examples can be generalized with the following definitions. Recall that $\mathcal{S}(X)$ denotes the set of all subpartitions for a set X.

Definition 5.7. *A partition $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ of V is called a C_4-**configuration (for** $G, \mathcal{P}, k$**)** if*

(i) $k \geq 2$ *is odd,*

(ii) $k > d_G(A_i)$ *for every* $i = 1, 2, 3, 4$,

(iii) $d_G(A_i, A_{i+2}) = 0$ *for every* $i = 1, 2$,

(iv) $k - d_G(A_i) + k - d_G(A_{i+2}) = \Phi$ *for both* $i = 1, 2$,

(v) there is a colour class $P \in \mathcal{P}$ and an index $i \in \{1, 2\}$ such that for $j = i$ and $j = i+2$ there exist subpartitions $\mathcal{X}_j \in \mathcal{S}(A_j)$ satisfying $\sum_{X \in \mathcal{X}_j} k - d_G(X) = k - d_G(A_j)$ and $\mathcal{X}_i \cup \mathcal{X}_{i+2} \in \mathcal{S}(P)$.

Definition 5.8. *A partition $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$ of V is called a C_6-**configuration (for** $G, \mathcal{P}, k$**)** if*

(i) $k \geq 2$ *is odd,*

(ii) $d_G(A_i) = k - 1$ *for every* $i = 1, 2, 3, 4, 5, 6$,

(iii) $d_G(A_i, A_{i+1}) = (k-1)/2$ *for every* $i = 1, 2, 3, 4, 5, 6$,

(iv) there exist three distinct colour classes $P_1, P_2, P_3 \in \mathcal{P}$ and subsets $A'_i \subseteq A_i$ satisfying $d_G('A_i) = k - 1$ for every $i = 1, 2, 3, 4, 5, 6$ such that $A'_j \cup A'_{j+3} \subseteq P_j$ for every $j = 1, 2, 3$.

With these definitions the solution of the minimum version of Problem 5.6 is the following.

Theorem 5.9 (Bang-Jensen, Gabow, Jordán and Szigeti [3]). *Let us be given a graph $G = (V, E)$, a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_r\}$ of the set V, and an integer $k \geq 2$. The minimum number of graph edges between the classes of $\mathcal{P}$ and making G k-edge-connected is Φ, unless a C_4 or a C_6 configuration exists, in which case it is $\Phi + 1$. Such an edge set can be found in polynomial time.*

The solution of the degree-specified problem is the following. We can formulate the following natural necessary conditions of the existence of a solution with degree function m if $k \geq 2$:

$$m(X) \geq k - d_{G_0}(X) \text{ holds for any nonempty } X \subsetneq V, \tag{5.6}$$

$$m(V) \text{ is even,} \tag{5.7}$$

$$m(P_i) \leq m(V - P_i) \text{ for every } i = 1, 2, \ldots, r. \tag{5.8}$$

However, similarly to the minimum version some exceptional cases show that these conditions are not sufficient. These exceptional cases are called **obstacles**: they are related to the configurations defined above, though there is an important difference. This is the following: it is possible that the optimum of the minimum version is Φ, but there is a degree-specification m satisfying (5.6)-(5.8) and yielding $m(V) = 2\Phi$, for which the degree-specified version is not solvable (because of an obstacle). This is illustrated in Figure 5.1.

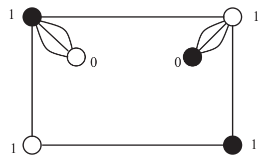

Figure 5.1: A C_4-obstacle. The minimum number of edges between black and white nodes making this graph 3-edge-connected is two, but the degree-specification given here does not yield a solution with 2 edges.

Let us give the definition of the obstacles. In the following 2 definitions we assume that m satisfies (5.6)-(5.8).

Definition 5.10. *A partition $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ of V is called a C_4-**obstacle (for** $G, \mathcal{P}, k, m$**)** if*

(i) $k \geq 2$ is odd,

(ii) $m(A_i) = k - d_G(A_i)$ for every $i = 1, 2, 3, 4$,

(iii) $d_G(A_i, A_{i+2}) = 0$ for every $i = 1, 2$,

(iv) $m(A_1 \cup A_3) = m(V)/2$ and the positive nodes of $A_1 \cup A_3$ all have the same colour.

Definition 5.11. *A partition $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$ of V is called a C_6-**obstacle** **(for** $G, \mathcal{P}, k, m$**)** if* *(i)* $k \geq 2$ *is odd,*

(ii) $m(A_i) = 1$ *and* $d_G(A_i) = k - 1$ *for every* $i = 1, 2, 3, 4, 5, 6$*,*

(iii) $d_G(A_i, A_{i+1}) = (k - 1)/2$ *for every* $i = 1, 2, 3, 4, 5, 6$*,*

(iv) the positive node in A_i *and in* A_{i+3} *have the same colour for every* $i = 1, 2, 3$*.*

The authors of [3] have shown that these are **the only obstacles** of the existence of a solution with degree function m.

Theorem 5.12 (Bang-Jensen, Gabow, Jordán and Szigeti [3])**.** *Let us be given a graph* $G_0 = (V, E_0)$*, a partition* $\mathcal{P} = \{P_1, P_2, \ldots, P_r\}$ *of the set* V*, and an integer* $k \geq 2$*. Assume that the degree-specification* $m \in \mathbb{Z}_+^V$ *satisfies* (5.6)-(5.8)*. Then there exists a graph* G *satisfying the degree-specification* m *and the partition constraints* $\mathcal{P}$ *such that* $G_0 + G$ *is* k*-edge-connected, unless* k *is odd and* G_0 *contains a* C_4*- or a* C_6*-obstacle.*

The main idea in the solution of the minimum version of Problem 5.6 is to find a degree-specification m satisfying (5.6)-(5.8) with $m(V)$ as small as possible, and trying to avoid creating a C_4- and a C_6-obstacle. The authors of [3] show that this is not possible if and only if the problem instance contains a **configuration**.

5.2 Preliminaries

In this section we give preliminaries about the methods that we will use in proving Theorem 5.4 and in solving the partition constrained version covering problem. We start with two simple observations about tight and dangerous sets. In many cases however we omit the reference to these lemmas, since the situation will be much simpler: for example $p \leq 1$ will hold.

Lemma 5.13. *Assume that* $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ *is a symmetric, positively crossing supermodular function and* $m \in C(p)$*. If* D *is a* p*-positive dangerous set and* T *is a tight set with* $m(D \cap T) > 0$ *then one of* D *and* T *contains the other.*

Proof. Certainly, T is also p-positive. They cannot satisfy $(-)$, since that would imply $m(D) - 1 + m(T) \leq p(D) + p(T) \leq p(D - T) + p(T - D) \leq m(D - T) + m(T - D) \leq m(D) - 1 + m(T) - 1$, a contradiction. This implies the lemma, since two p-positive sets always satisfy $(-)$ unless one contains the other. ◻

Lemma 5.14. *Assume that* $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ *is a symmetric, positively crossing supermodular function and* $m \in C(p)$*. If a* p*-positive dangerous set* D *and a* p*-positive tight set* T *cross each other then* $D \cup T$ *and* $D \cap T$ *is also dangerous.*

Proof. Assume that one of $D \cup T$ and $D \cap T$ is not dangerous and apply $(\cap\cup)$ for D and T to get $m(D)-1+m(T) \leq p(D)+p(T) \leq p(D\cap T)+p(D\cup T) \leq m(D\cap T)+m(D\cup T)-2$, a contradiction. □

The following simple lemma due to Benczúr and Frank follows easily from Claim 1.9.

Lemma 5.15. *If $p\colon 2^V \to \mathbb{Z}\cup\{-\infty\}$ is a symmetric, positively crossing supermodular function and $\{X_1, X_2, \dots, X_t\}$ is a partition of V satisfying $p(X_1)=1$ and $p(X_1 \cup X_i) > 0$ for any $i = 1, 2, \dots, t$, then this partition is p-full.* □

However we will need the following, slightly more complicated lemma.

Lemma 5.16. *Let $p\colon 2^V \to \mathbb{Z}\cup\{-\infty\}$ be a symmetric, positively crossing supermodular function and $\{V_1, V_2, \dots, V_k\}$ be a partition of V satisfying $p(V_i)=1$ for any $i=1,2,\dots,k$ (where $k \geq 4$). Let furthermore $U_i^1, U_i^2, \dots, U_i^{t_i}$ be a partition of V_i (where $t_i \geq 1$ is an integer) for any $i=1,2,\dots,k$ such that $p(V_i \cup U_j^l) > 0$ for any possible i, j, l. Assume furthermore that $p(U_1^1) = 1$. Then the partition $\mathcal{U} = \{U_i^j : i = 1, 2, \dots, k$ and $j = 1, 2, \dots, t_i\}$ is p-full.*

Proof. Let $i \in \{1,2,\dots,k\}$ be arbitrary and $\mathcal{U}' \subseteq \mathcal{U}$ such that $V_i \cup (\bigcup \mathcal{U}') \neq V$. By Claim 1.9, $p(V_i \cup (\bigcup \mathcal{U}')) > 0$. Applying this and the symmetry of p we get that $p(U_1^1 \cup U_j^l) = p(V - (U_1^1 \cup U_j^l)) > 0$ for any possible j, l (choose an arbitrary $V_i \subseteq V - (U_1^1 \cup U_j^l)$ and an appropriate $\mathcal{U}' \subseteq \mathcal{U}$). But then we can apply Lemma 5.15 in order to finish this proof. □

Our approach in the algorithms to be given in this chapter is the following: we perform some splitting-off steps, and sometimes we undo one splitting-off in order to make a progress. Therefore we introduce the following operations. Let $p_0\colon 2^V \to \mathbb{Z}\cup\{-\infty\}$ be symmetric, positively crossing supermodular and $m_0 \in C(p_0) \cap \mathbb{Z}^V$ with $m_0(V)$ even. Assume that we have performed some admissible splitting-offs and let G be the graph of the split edges. Let furthermore $p = p_0 - d_G$ and $m(v) = m_0(v) - d_G(v)$ for every $v \in V$. Pick an edge $uv = e \in G$. Recall that the **unsplitting** operation of e is simply the reverse of the splitting-off operation: $m^e = m + \chi_{\{u\}} + \chi_{\{v\}}$, $G^e = G - e$ and $p^e = p + d_{(V,\{(uv)\})} = p_0 - d_{G^e}$. Of course, this is always admissible, that is $m^e \in C(p^e)$.

In the sequel, we will use the following operation. Let $uv \in E(G)$ and $x \in V^+$. The **edge-switch operation (at x, u, v)** is the following: let $G' = G - uv + xu$, let $m' = m - \chi_{\{x\}} + \chi_{\{v\}}$ and $p' = p_0 - d_{G'}$ (i.e. unsplit uv and split-off at x, u: the edge xu will also be called a split edge). If $m' \in C(p')$ then the edge-switch is **admissible**. If there is a set $X \subseteq V$ with $0 = p(X) = m(X) - 1$, $x, u \in X$ but $v \notin X$ then the edge-switch at x, u, v is clearly not admissible: we will say that such a set is a (x, u, v)**-switchblocking set**. An

(x, uv)-**switchblocking set** is a set which is either an (x, u, v)-switchblocking set, or an (x, v, u)-switchblocking set.

If x and y are two (distinct) positive nodes and uv is an edge of G then a **one-change (at x, u, v, y)** is the following operation: $m' = m - \chi_{\{x\}} - \chi_{\{y\}}$, $G' = G - e + ux + vy$ and $p' = p_0 - d_{G'}$ (i.e. unsplit uv and split-off at x, u and at y, v: the edges xu and yv will also be called *split edges*). It will be an **admissible one-change** if $m' \in C(p')$. Note that if the edge-switch at x, u, v or the one at y, v, u is not admissible (for example because there exists an (x, u, v)-switchblocking set, or an (y, v, u)-switchblocking set), then the one-change at x, u, v, y is not admissible.

A partition of V into p-positive tight sets will be called a p-**tight partition**. In the arguments below after performing one of the above operations we will usually (unless stated otherwise) replace the functions m and p by the modified functions. The following statements will be useful in the proofs of this chapter.

Claim 5.17. *Let $p\colon 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric, positively crossing supermodular function and $m \in C(p) \cap \mathbb{Z}^V$. Assume that $p \le 1$. Then a set $X \subseteq V$ having $p(X) = 1$ cannot cross a p-positive tight set Y.*

Proof. Assume that X crosses Y. Note that $p(Y) = m(Y) = 1$. By possibly complementing X we can assume that $m(Y \cap X) = 0$. But $(\cap\cup)$ for X and Y implies that $p(Y \cap X) = 1$, a contradiction with $m \in C(p)$. □

Lemma 5.18. *Let $p_0\colon 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric, positively crossing supermodular function, G be a graph and $p = p_0 - d_G$ and $m \in C(p) \cap \mathbb{Z}^V$. Let x and y be two (distinct) positive nodes and uv be an edge of G. The edge-switch at x, u, v is not admissible if and only if there exists a set X containing x and u but excluding v which is dangerous with respect to p and m. If the edge-switch at x, u, v and the one at y, v, u are both admissible then the one-change at x, u, v, y is not admissible if and only if there exists a set X containing x, u, v, y which is dangerous with respect to p and m.*

Proof. The first statement is clear: since the unsplitting at u and v is always admissible, the edge-switch x, u, v is not admissible if and only if there exists a set X that is dangerous containing x and u after this unsplitting (i.e. with respect to p^{uv} and m^{uv}). If this set also contains v then $m(X) - p(X)$ does not change during the edge-switch, so X does not become deficient. Therefore X does not contain v, therefore $m(X) - p(X)$ will decrease by two after the edge-switch, so it becomes deficient if and only if it was dangerous with respect to p and m.

To prove the second statement observe that the one-change at x, u, v, y can be considered as the sequence of a one-change at x, u, v followed by a splitting-off at y, v. Let m' and p'

be the functions after the edge-switch at x, u, v. If the splitting is not admissible after this edge-switch then there is an X containing y and v which is dangerous with respect to p' and m'. Note that this X is not dangerous with respect to p^{uv} and m^{uv} (this is because the edge-switch y, v, u was also admissible originally), $m^{uv}(X) - p^{uv}(X) > m'(X) - p'(X)$ must hold, in other words X must contain x and u. But then $m(X) - p(X) = m'(X) - p'(X)$, therefore X was also dangerous with respect to p and m. □

We will only apply the operations introduced above under special circumstances. The next lemma describes this. Note that if $p \leq 1$ then a dangerous set containing a positive node will have p-value 0 or 1.

Lemma 5.19. *Let $p_0 \colon 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric, positively crossing supermodular function, G be a graph and $p = p_0 - d_G$ and $m \in C(p) \cap \mathbb{Z}^V$. Assume that $p \leq 1$ and $\{V_1, V_2, \ldots, V_k\}$ is a p-tight partition of V (i.e. $m(V_j) = p(V_j) = 1$ for every $j = 1, 2, \ldots, k$), where $k \geq 4$, and let $v_j \in V_j \cap V^+$ for every $j \in \{1, 2, \ldots, k\}$. Then*

1. *the tight partition $\{V_1, V_2, \ldots, V_k\}$ is unique,*
2. *if $uv \in E(G)$ is induced in V_i for some $i \in \{1, 2, \ldots, k\}$ and $v_r, v_s \in V^+ - v_i$ then*
   - (a) *if X is a (v_r, u, v)-switchblocking set, then $X - V_i = V_r$ and there is no (v_r, v, u)-switchblocking set,*
   - (b) *if X is a (v_r, u, v)-switchblocking set, and Y is a (v_s, u, v)-switchblocking set then $X - Y = V_r$ and $Y - X = V_s$,*
   - (c) *the edge-switch at v_r, u, v is not admissible if and only if there exists a (v_r, u, v)-switchblocking set,*
   - (d) *the one-change at v_r, u, v, v_s is not admissible if and only if there exists a (v_r, u, v)-switchblocking set, or a (v_s, v, u)-switchblocking set,*
3. *if $uv \in E(G)$ is such that $u \in V_1$ and $v \in V_2$ then*
   - (a) *the edge-switch at v_1, v, u is always admissible,*
   - (b) *the edge-switch at v_3, u, v is not admissible if and only if $p(V_1 \cup V_3) = 1$,*
   - (c) *the one-change at v_3, u, v, v_1 is not admissible if and only if $p(V_1 \cup V_3) = 1$.*

Proof. To prove the first statement, suppose that the partition $\{V_1', V_2', \ldots, V_k'\}$ also satisfies that $m(V_i') = p(V_i') = 1$ for every $i = 1, 2, \ldots, k$ (assume that these sets are indexed such that $m(V_i \cap V_i') = 1$ for every $i = 1, 2, \ldots, k$). Since V_i' cannot cross V_i for all i, and $V_i \subsetneq V_i'$

would imply that V_i' crosses another V_j (since $k > 2$), this means that $V_i = V_i'$ for every $i = 1, 2, \ldots, k$.

To prove (2a) apply $(\cap\cup)$ for X and V_i to get that $p(X \cup V_i) = 1$, implying that $X - V_i = V_r$ by Claim 5.17. If X is a (v_r, u, v)-switchblocking set and Y is a (v_r, v, u)-switchblocking set then (1.5) for X and Y gives that $p(X \cup Y) = 1$, contradicting Claim 5.17.

To show (2b) apply (1.6) for X and Y.

To prove (2c) observe that if the edge-switch at v_r, u, v is not admissible then by Lemma 5.18 there is a dangerous set X with $v_r, u \in X$ and $v \notin X$. Since $m(V) \geq 4$, this set crosses V_i, therefore it is a (v_r, u, v)-switchblocking set, as claimed.

The proof of (2d) is similar: if the edge-switch at v_r, u, v or the one at v_s, v, u is not admissible then this is because of a switchblocking set by (2c). If both of these edge-switch operations are admissible but the one-change at v_r, u, v, v_s is not admissible then by Lemma 5.18 there exists a dangerous set X containing v_r, u, v, v_s. Since $m(X) \geq 2$ by the nodes $v_r, v_s \in X$, this is only possible if $m(X) = 2 = p(X) + 1$. But this X then crosses V_i, contradicting Claim 5.17.

To prove (3a) assume that the edge-switch at (v_1, v, u) is not admissible. Use Lemma 5.18 and note that the dangerous set X will cross V_1, so it cannot have p-value 1, i.e. it is a (v_1, v, u)-switchblocking set. Apply $(\cap\cup)$ for X and V_2 to get that $p(X \cup V_2) = 1$, contradicting Claim 5.17.

To prove (3b) assume that the edge-switch at v_3, u, v is not admissible. By Lemma 5.18 there is a dangerous set X containing v_3, u and not containing v. Assume that $p(X) = 0$. Then X crosses V_1 (since $v_1 \notin X$), therefore applying (1.6) to X and V_1 gives a contradiction. Therefore $p(X) = 1$, and then by Claim 5.17 $X = V_1 \cup V_3$.

To prove (3c) assume that the one-change at v_3, u, v, v_1 is not admissible but $p(V_1 \cup V_3) < 1$, i.e. the edge-switch at (v_3, u, v) is admissible. By Lemma 5.18 there exists a dangerous set X containing v_3, u, v, v_1, thus $m(X) = 2 = p(X) - 1$ and X crosses V_2, since $v_2 \notin X$, a contradiction by Claim 5.17. $\square$

5.3 Proof of the theorem of Benczúr and Frank

In this section we will give an algorithmic proof of Theorem 5.4. This proof appeared in [8]. The idea is the following: first we perform an arbitrary sequence of admissible splitting-offs as long as possible. When there is no more admissible splitting-off, then we try to unsplit **one single edge** such that we obtain two admissible splitting-offs (this operation was called an *admissible one-change*). Surprisingly, this will be enough: we prove that this algorithm finds the required degree-specified graph, if the conditions of Theorem 5.4 hold.

Proof of Theorem 5.4. The necessity of the conditions is clear. The proof of sufficiency uses the algorithm sketched above. In fact we will prove a little more, as in this proof the algorithm takes as input the function p_0 and an arbitrary $m_0 \in C(p_0) \cap \mathbb{Z}^V$ with $m_0(V)$ even (that is, we don't assume about the input that $m_0(V)/2 \geq dim(p_0)-1$ holds). We will prove that if this algorithm does not terminate with finding the required degree-specified graph then the condition $m_0(V)/2 \geq dim(p_0) - 1$ did not hold. In this case we will also find a p_0-full partition of size $dim(p_0)$. In fact we will prove that the algorithm finds a longest possible admissible splitting sequence. Let us formally describe the algorithm.

First Step of the Algorithm: Perform an arbitrary sequence of admissible splitting steps as long as there exists one.

Before continuing the description of the algorithm we describe the situation when there is no further admissible splitting-off. As usual, let the graph of the edges split so far be denoted by G, $p = p_0 - d_G$ and $m(v) = m_0(v) - d^+_G(v)$ for any $v \in V$. By Lemma 4.6, $p \leq 1$ and any pair $u, v \in V^+$ is in a dangerous set X: this means that $p(X) = 1$ and $m(X) = 2$, hence $m \leq 1$. We can further assume that $m(V) \geq 4$ (if $m(V) = 0$ then the theorem is proved and clearly $m(V) = 2$ cannot be the case). Let the positive nodes be $v_1, v_2, \ldots, v_k$ (where $k = m(V)$ is of course even).

Lemma 5.20. *Under the assumptions made above*

(i) for any $i \in \{1, 2, \ldots, k\}$ there exists a unique maximal tight set V_i containing v_i,

(ii) the set blocking v_i and v_j is $V_i \cup V_j$ for any $i, j \in \{1, 2, \ldots, k\}$,

(iii) $p(\cup_{i\in I} V_i) = 1$ for any nonempty $I \subsetneq \{1, 2, \ldots, k\}$,

(iv) the sets $V_1, V_2, \ldots, V_k$ form a partition of V.

Proof. The sets that we consider will always have positive p value, so we can use $(\cap\cup)$ and $(-)$ if two of them cross. A set X blocking a pair $u, v \in V^+$ and another set Y blocking a pair $w, v \in V^+$ cross each other, meaning that $p(X \cap Y) = 1$, so $v \in V^+$ is in a tight set. If T_1 and T_2 are two tight sets containing the positive node $x \in V^+$ then T_1 and T_2 cannot cross each other, since then $(-)$ would imply that $p(T_1 - T_2) = 1 > m(T_1 - T_2) = 0$, a contradiction. Thus one of T_1 and T_2 must contain the other, so indeed there exists a unique maximal tight set V_i containing v_i for every i.

To prove (ii) let i, j be two different indices between 1 and k. It is straightforward that V_i and V_j have to be disjoint (otherwise $p(V_i \cap V_j) = 1$ would follow from $(\cap\cup)$). Similarly, a set X blocking v_i and v_j must contain V_i (and V_j), otherwise $p(V_i - X) = 1$ would follow from $(-)$. On the other hand, if $l \in \{1, 2, \dots, k\}$ is different from i and j and Y is a set blocking v_i and v_l then $(\cap\cup)$ implies that $X \cap Y = V_i$ (since it is tight) and $(-)$ implies that $X - Y = V_j$ (since it is tight again). This finishes the proof of (ii), and then (iii) follows from Claim 1.9.

The only thing to be proved to get (iv) is that $\cup_{i=1}^k V_i = V$: but if this was not the case then Claim 1.9 would also imply that $p(\cup_{i=1}^k V_i) = 1$, which would give a contradiction, since $m(V - \cup_{i=1}^k V_i) = 0$ and $p(V - \cup_{i=1}^k V_i) = p(\cup_{i=1}^k V_i) = 1$.

One simple observation shows that G does not have edges between two classes V_i and V_j of this partition: if it had, then choosing a third index $l \in \{1, 2, \dots, k\}$ and using that $X = V_i \cup V_l$ and $Y = V_j \cup V_l$ has to satisfy $(\cap\cup)$ with equality would give a contradiction by (1.1).

The next step of the algorithm tries to find admissible one-changes as long as possible. That is, our aim is to find an edge $uv = e$ of G (spanned by V_i, say) and two positive nodes v_r and v_s such that an admissible one-change can be performed. Note that $r = i$ is not a good choice, since $p(V_i \cup V_s) = 1$; $s = i$ is not good, either, so v_r and v_s are both distinct from v_i. By Lemma 5.19 (2d), the obstacle of the admissibility of the one-change at v_r, u, v, v_s is a switchblocking set. The following lemma tells us that if a one-change is not admissible at v_r, u, v, v_s for some edge $uv \in E(G)$, then there is no admissible one-change at this edge at all.

Claim 5.21. *If X is a (v_r, uv)-switchblocking set (where $uv \in E(G)$ is induced by V_i for some i and $v_r \in V^+ - v_i$), then $e = \Delta_G(X)$ and $V_j \cup (X \cap V_i)$ is the unique (v_j, uv)-switchblocking set for any $j \neq i$.*

Proof. Without loss of generality assume that X is a (v_r, u, v)-switchblocking set. We will use that $m(V) \geq 4$: let i, r be the indices of the statement and let $q, s \in \{1, 2, \dots, k\} - \{i, r\}$ be distinct indices. Apply $(\cap\cup)$ to X and $Y = V_r \cup V_q$ to get that $p(X \cup Y) \geq 0$, but since it cannot be 1 by Claim 5.17, it must be 0. Now apply $(-)$ to $X \cup Y$ and $V_r \cup V_s$ to obtain that $p(V_q \cup (X \cap V_i)) \geq 0$, but again by Claim 5.17 it cannot be one, so $V_q \cup (X \cap V_i)$ is indeed a (v_q, uv)-switchblocking set for any $q \neq i$. Apply (1.6) for X and $V_q \cup (X \cap V_i)$ to get that $e = \Delta_G(X)$. Finally Lemma 5.19 (2b) gives the uniqueness of (v_j, u, v)-switchblocking set for any $j \neq i$.

After these preliminaries the description of the algorithm is continued.

Second Step of the Algorithm: Perform an arbitrary sequence of admissible one-change operations as long as there exists one. If $m(V)$ decreases to 2 then finish the procedure with a splitting-off (which is of course admissible) and terminate.

Observe that an admissible one-change will not create an admissible splitting-off by Lemma 5.20 (iii), unless $m(V)$ becomes 2.

Let us now describe the situation when there is no further admissible one-change.

For simplicity let us denote the remaining degree specification again by m, the obtained graph by G and $p = p_0 - d_G$. Again, if $m(V) \leq 3$ then we are done, so assume that $m(V) \geq 4$. We will now show how to obtain a p_0-full partition of size greater than $m_0(V)/2 + 1$, which finishes the proof of the theorem. Furthermore this partition will be of size $m(V) + |E(G)|$, which shows that the dimension of p_0 is not greater than this, since adding a spanning tree on V^+ to G covers p_0 and has size $m(V) + |E(G)| - 1$. To this end let us describe the structure of the obstacles of further admissible one-changes. By Claim 5.21 for every split edge e induced in V_i, say, there exists a uniquely defined set $X_e \subseteq V_i - v_i$ such that $X_e \cup V_j$ is the unique (v_j, e)-switchblocking set for any $j \neq i$. Since $d_G(X_e) = 1$ for any edge, this implies that G does not contain a cycle.

Claim 5.22. *Let e, f be two (distinct) edges of G spanned by V_i. Then X_e and X_f cannot cross each other.*

Proof. Assume the contrary and consider two positive nodes v_r and v_s (distinct from v_i). Apply $(-)$ for the crossing sets $X = V_r \cup X_e$ and $Y = V_s \cup X_f$ to get that $p(X - Y) = p(Y - X) = 0$. Since at least one of $X - Y$ and $Y - X$ is entered by at least one of e and f, this contradicts the uniqueness of the sets X_e and X_f.

This claim shows that the sets $\{X_e : e \in G\}$ form a laminar family. Visually, if we introduce an orientation $\vec{G}$ of G such that an edge $e \in G$ is oriented to enter X_e, then we get that the undirected components of $\vec{G}$ are arborescences and the laminar family $\{X_e : e \in G\}$ can naturally be related to these arborescences (it is known that an arborescence naturally defines a laminar family), though we don't need this observation below.

Consider the laminar family $\mathcal{U} = \{V_1, V_2, \ldots, V_k\} \bigcup \{X_e : e \in G\}$ and let $U^* = U - \bigcup_{W \in \mathcal{U}, W \subsetneq U} W$ for any $U \in \mathcal{U}$. The following claim finishes the proof of the theorem, since the family $\{U^* : U \in \mathcal{U}\}$ has $m(V) + |E(G)|$ members.

Claim 5.23. *The set U^* is not empty for any $U \in \mathcal{U}$. The partition $\{U^* : U \in \mathcal{U}\}$ is p_0-full.*

Proof. Observe that U^* is never empty, since if $U = V_i$ for some i then $v_i \in U^*$, and if $U = X_e$ then one endpoint of e is in U^*. Now we want to show that $p_0(V_i \cup U^*) > 0$ for any $i \in \{1, 2, \ldots, k\}$ and $U \in \mathcal{U}$. We can assume that $U \subseteq V_j$ where $j \neq i$ and let $l \in \{1, 2, \ldots, k\} - \{i, j\}$. Apply Claim 1.9 for V_l and the maximal sets of the family

$\{W \in \mathcal{U} : W \subsetneq U\}$ to deduce that $p_0(V_l \cup X) > 0$ where $X = \bigcup_{W \in \mathcal{U}, W \subsetneq U} W$. Now $(-)$ for $V_i \cup U$ and $V_l \cup X$ gives that $p_0(V_i \cup U^*) \geq p_0(V_i \cup U) + p_0(V_l \cup X) - p_0(V_l) \geq 1+1-1 \geq 1$, as claimed. Now Lemma 5.16 finishes the proof of this claim, since if U is a minimal member of the family $\mathcal{U}$ then $p_0(U) = p_0(U^*) = 1$ can be easily verified.

□

Let us give the pseudocode of the algorithm that we suggest (for simplicity we do not read out the p-full partition of size $dim(p)$ as in the proof above). We note that by the arguments given after Algorithm GREEDYCOVER, this algorithm can also be implemented to run in polynomial time.

Algorithm SYMCROS_COVER
begin

> **INPUT**: A **symmetric, positively crossing supermodular** function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ (given with a maximizing oracle) and an admissible degree specification $m \colon V \to \mathbb{Z}_+$ (with $m(V)$ even).
>
> **OUTPUT**: A graph $G = (V, E)$ covering p and satisfying $d^+_G(v) = m(v)$ for every $v \in V$, or the statement that the condition $m(V)/2 \geq dim(p) - 1$ did not hold.

1.1. Initialize $G = (V, \emptyset)$.
1.2. While there exists an admissible splitting-off at u, v, perform it, i.e.
1.3. Let $m = m - \chi(u) - \chi(v)$ and $p = p - d_{(V,\{(u,v)\})}$ and $G = G + (uv)$.
1.4. EndWhile
1.5. While there exists an admissible one-change at x, u, v, y, perform it, i.e.
1.6. Let $m = m - \chi(x) - \chi(y)$, $G = G - uv + xu + vy$ and $p = p + d_{(V,\{uv\})} - d_{(V,\{xu,vy\})}$.
If $m(V) = 2$ then finish with an admissible splitting-off, i.e.
1.7. Let $m = m - \chi(u) - \chi(v)$ (where $V^+ = \{u, v\}$) and $p = p - d_{(V,\{(u,v)\})}$ and $G = G + (uv)$.
1.8. EndWhile
1.9. If $m(V) = 0$ then output G.
1.10. Otherwise return "The condition $m(V)/2 \geq dim(p) - 1$ did not hold!".
end

In fact our proof implies the following deficient form of Theorem 5.4 (where an *admissible splitting sequence* means an arbitrary sequence of admissible splitting-offs).

Theorem 5.24. *Let $p_0 \colon 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric, positively crossing supermodular set function and $m_0 \in C(p_0) \cap \mathbb{Z}^V$ with $m_0(V)$ even. If $m_0(V)/2 < \dim(p_0) - 1$ then the longest admissible splitting sequence consists of $m_0(V) - \dim(p_0)$ splitting-offs. If $m_0(V)/2 \geq \dim(p_0) - 1$ then there exists a complete admissible splitting-off.*

Proof. Consider an arbitrary running of the algorithm sketched above. If it gets stuck with remaining degree specification m and graph G, then we have seen that $m(V) + |E(G)| = \dim(p_0)$. Since $m(V) + 2|E(G)| = m_0(V)$, this shows that (after arbitrary choices in the algorithm) $|E(G)| = m_0(V) - \dim(p_0)$. Since a longest admissible splitting sequence is clearly a valid running of the algorithm (there cannot exist an admissible one-change after a longest splitting sequence), this finishes the proof. $\square$

5.4 Partition constrained covering problem

In this section we will also consider **partition (or colour) constraints**. That is we are also given a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_r\}$ of the set V (colour classes) and we are only allowed to introduce edges **between** two members of this partition. If $v \in P_i$ for some $v \in V$ and i then we will also use the notation $c(v) = i$ (that is, v **has colour** i). We formulate the problems that we want to solve in this section.

Problem 5.25 (Partition constrained covering of a symmetric, positively crossing supermodular function)**.** *Let us be given a symmetric, positively crossing supermodular function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ with a maximizing oracle and a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_r\}$ of the set V. The problem is to find a graph G covering p that contains **only edges between the classes** of $\mathcal{P}$. In the **minimum version** the number of edges of G is to be minimized. In the **degree-specified version** G has to satisfy a given degree-specification $m \in \mathbb{Z}_+^V$.*

Note that loop edges in G are prohibited by the partition constraints, so $d_G^+(v) = d_G(v)$ for any $v \in V$.

This section contains joint results with Roland Grappe and Zoltán Szigeti. These results are not published yet. They have been accepted for presentation at the ACM-SIAM Symposium on Discrete Algorithms (SODA10) which will be held in Austin, Texas in January 2010: see [10].

5.4.1 Preliminaries

Lower bounds and necessary conditions

In this subsection we give natural lower bounds for the minimum version of Problem 5.25 and natural necessary conditions for the degree-specified version of that problem. We start with the latter.

Assume that the degree-specified version of Problem 5.25 is given. We have seen that a natural necessary condition of the existence of a solution is that the degree-specification is admissible, that is

$$m(X) \geq p(X) \text{ for any } X \subseteq V. \tag{5.9}$$

It is easy to see that the following conditions are also necessary.

$$m(V) \text{ is even,} \tag{5.10}$$

$$m(P_i) \leq m(V - P_i) \text{ for every } i = 1, 2, \ldots, r. \tag{5.11}$$

We will say that a degree-specification m satisfying (5.9)-(5.11) is **allowed**. A further necessary condition of the existence of a solution with degree function m is that

$$m(V)/2 \geq dim(p) - 1. \tag{5.12}$$

We emphasize that an allowed degree-specification does not necessarily satisfy the dimension condition (5.12).

However the conditions (5.9)-(5.12) are still not sufficient, there are some exceptional cases when this graph does not exist as this was already seen at the problem of partition constrained global edge-connectivity augmentation. These exceptions will be called **obstacles** here, too. We will give the definition of these obstacles in Section 5.4.2.

Turning to the minimum version of the problem: by the previous arguments the value $\phi = \max\{\alpha_p, \beta_p^1, \ldots, \beta_p^r, \dim(p) - 1\}$ is a lower bound on the number of edges of the desired graph, where

$$\alpha_p = \lceil SLB(p)/2 \rceil = \max\{\lceil \frac{\sum_{X \in \mathcal{X}} p(X)}{2} \rceil : \mathcal{X} \text{ is a subpartition of } V\}$$

$$\beta_p^i = \max\{\sum_{Y \in \mathcal{F}} p(Y), \mathcal{F} \text{ subpartition of } P_i\}, \text{for } i = 1, \ldots r.$$

Again, as seen at the problem of partition constrained global edge-connectivity augmentation, in some cases this lower bound cannot be achieved. These exceptional cases will be called **configurations**, their description will be given in Section 5.4.3. Since obstacles and configurations are closely related to each other, they have a common root, we give the description of this common root in the next section. We call them **constructions**: they express certain properties of the set function p, i.e. the partition $\mathcal{P}$ does not play a role in the definition of constructions.

Constructions

We will use the following convention. If $A_1, A_2 \ldots, A_l$ are some subsets of V, then A_{l+1} will denote A_1, A_{l+2} will be A_2 and so on (and similarly A_0 will be A_l). Furthermore, a construction (an obstacle, or a configuration, see later) will always be a partition $\mathcal{A} = \{X_1, X_2, \ldots, X_t\}$ of V: though the order of the partition classes does matter in some sense in these definitions, we decided to use the set-notation $\{X_1, X_2, \ldots, X_t\}$ instead of the sequence-notation $(X_1, X_2, \ldots, X_t)$, since some reordering will always be possible. Hopefully after this remark this will not cause a confusion. Recall that $m \in C(p)$ is said to be minimal if $m(V) = SLB(p) = \max\{\sum_{X\in\mathcal{X}} p(X) : \mathcal{X}$ is a subpartition of $V\}$.

Definition 5.26. *A partition $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ of V is called a **C_4^*-construction for p** if*

1. $p(A_i) + p(A_{i+1}) - p(A_i \cup A_{i+1})$ *is odd for every* $i = 1, 2, 3, 4$,
2. $p(A_{i-1} \cup A_i) + p(A_i \cup A_{i+1}) = p(A_{i-1}) + p(A_{i+1})$ *for every* $i = 1, 2, 3, 4$,
3. $p(A_1 \cup A_3) \le 0$ *and* $p(A_2 \cup A_4) \le 0$,
4. $p(A_1) + p(A_3) = p(A_2) + p(A_4) = \frac{1}{2}SLB(p)$.

Note that the order of the partition classes in a C_4^*-construction matters only in the following sense: if $\{A_1, A_2, A_3, A_4\}$ is a C_4^*-construction, then so is $\{A_2, A_3, A_4, A_1\}$ (**cyclical reindexing**) and $\{A_4, A_3, A_2, A_1\}$ (**reversing**).

The following structure arises only for our abstract form of the problem, it does not exist in the framework of graphs or hypergraphs.

Definition 5.27. *A partition $\mathcal{A} = \{A_1, A_2, A_3, A_4, B_1, \ldots, B_t\}$ (where $t \ge 1$) of V is called a **C_5^*-construction for p** if*

1. $p(A_i) = 1$ *for every* $i = 1, 2, 3, 4$,
2. $p(B_j) = 2$ *for every* $j = 1, \ldots, t$,
3. $p(A_i \cup B_j) = p(A_i \cup A_{i+1}) = 1$ *for every* $i = 1, 2, 3, 4$ *and* $j \in \{1, \ldots, t\}$,
4. $p(A_1 \cup A_3) \le 0$ *and* $p(A_2 \cup A_4) \le 0$,
5. $SLB(p) = \sum_{X\in\mathcal{A}} p(X) = 2t + 4$.

Observe that we can cyclically reindex the classes A_i, or reverse the order of them in a C_5^*-construction, while the classes B_j can come in any order.

Definition 5.28. *A partition $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$ of V is called a **C_6^*-construction for p** if*

1. $p(A_i) = 1$ *for every* $i = 1, 2, 3, 4, 5, 6$,

2. $p(A_i \cup A_{i+1}) = 1$ *for every* $i = 1, 2, 3, 4, 5, 6$,

3. $p(A_i \cup A_j) \leq 0$ *for every* $1 \leq i \leq 4$ *and* $i + 2 \leq j \leq i + 4$,

4. $SLB(p) = \sum_{i=1}^{6} p(A_i) = 6$.

Again note that we can cyclically reindex the classes, or reverse the order of the classes in a C_6^*-construction.

We say that a partition $\mathcal{A}$ **is a construction** if it is either a C_4^*-, a C_5^*- or a C_6^*-construction. Note, that if $\mathcal{A}$ is a construction and $m \in C(p)$, then the condition $m(V) = SLB(p)$ (minimality of m) is equivalent to saying that every $X \in \mathcal{A}$ is tight. Also note that if $\mathcal{A}$ is a C_4^*-construction and $m \in C(p)$ is minimal then Definition 5.26.1 requires that $m(A_i \cup A_{i+1}) - p(A_i \cup A_{i+1})$ has to be odd for every $i = 1, 2, 3, 4$.

Claim 5.29. *If* $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ *is a* C_4^**-construction for* p *then* $p(A_i) > 0$ *for every* $i = 1, 2, 3, 4$. *Consequently, a set containing at least three members of* $\mathcal{A}$ *cannot be dangerous.*

Proof. Assume for example that $p(A_2) = 0$ (it cannot be negative since that would imply $SLB(p) = \sum_i p(A_i) < \sum_{i \neq 2} p(A_i) \leq SLB(p)$) and let $m \in C(p)$ be minimal. Definition 5.26.2 applied to $i = 2$, the tightness of A_1, A_2 and A_3, and the admissibility of m gives that $p(A_1 \cup A_2) + p(A_2 \cup A_3) = p(A_1) + p(A_3) = m(A_1) + m(A_3) = m(A_1 \cup A_2) + m(A_2 \cup A_3) \geq p(A_1 \cup A_2) + p(A_2 \cup A_3)$, implying that $A_1 \cup A_2$ and $A_2 \cup A_3$ are both tight, contradicting Definition 5.26.1.

To get the corollary assume that a set X containing $A_1 \cup A_2 \cup A_3$ is dangerous. Since $p(X) = p(V - X)$, this implies that $SLB(p) \geq p(X) + p(V - X) = 2p(X) \geq 2(m(X) - 1) \geq 2(SLB(p)/2 + 1 - 1)$, i.e. $p(X) = SLB(p)/2$, but $m(V - X) \leq SLB(p)/2 - 1$, a contradiction. □

Claim 5.30. *Let* $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ *be a* C_4^**-construction,* $m \in C(p)$ *be minimal and* $i \in \{1, 2, 3, 4\}$ *arbitrary. If* $A_i \cup A_{i+1}$ *is not dangerous then a pair* $x \in A_i \cap V^+$ *and* $y \in A_{i+1} \cap V^+$ *is admissible. If we split-off this pair then* $\mathcal{A}$ *becomes a* C_4^**-construction for the modified function* $p' = p - d_{(V,\{xy\})}$.

Proof. Without loss of generality let $i = 1$. Assume that the pair x, y is not admissible and let X be a dangerous set blocking it. We can assume that X is a maximal dangerous set, thus by Lemma 5.13 $A_1 \cup A_2 \subseteq X$. Similarly, if for example $A_3 \cap X \neq \emptyset$ then A_3 and X would cross each other (since $A_4 \nsubseteq X$ by Claim 5.29), and then Lemma 5.14 would imply that $A_3 \subseteq X$, contradicting Claim 5.29. Therefore $X = A_1 \cup A_2$, which is not dangerous by the assumption. The second statement is easy to check (to get Definition 5.26.4 use Lemma 2.16). □

The following properties of a C_5^*-construction can be proved.

Claim 5.31. *If $\mathcal{A} = \{A_1, A_2, A_3, A_4, B_1, \dots, B_t\}$ is a C_5^*-construction and $m \in C(p)$ is arbitrary with $m(V) = SLB(p)$ then*

(i) $p(A_i \cup \bigcup_{j\in J} B_j) = p(A_i \cup A_{i+1} \cup \bigcup_{j\in J} B_j) = 1$ *for all* $J \subseteq \{1, \dots, t\}$ *and* $i \in \{1, 2, 3, 4\}$,

(ii) $p(\cup_{j\in J} B_j) = 2$ *for all nonempty* $J \subseteq \{1, \dots, t\}$,

(iii) $p(A_i \cup A_{i+2} \cup \bigcup_{j\in J} B_j) = 0$ *for all* $J \subseteq \{1, \dots, t\}$ *and* $i \in \{1, 2\}$

(iv) *If X crosses a member Y of the partition $\mathcal{A}$ then $p(X) \le p(X \cup Y)$ and $p(X) \le p(X - Y)$. If furthermore $m(X \cap Y) < m(Y)$ then $p(X) < p(X \cup Y)$, and similarly if $m(Y - X) < m(Y)$ then $p(X) < p(X - Y)$.*

(v) *If X crosses a member of the partition $\mathcal{A}$ then $p(X) \le 1$ (consequently $p \le 2$),*

(vi) *the splitting at u and v is admissible for any positive pair $u \in A_i \cup B_{j_1}$ and $v \in A_{i+2} \cup B_{j_2}$ (where $i \in \{1, 2, 3, 4\}$ and $j_1, j_2 \in \{1, 2, \dots, t\}$ are distinct),*

(vii) *Splitting-off at $a \in A_1 \cap V^+$ and $b \in B_1 \cap V^+$ the partition $\{A_1 \cup B_1, A_2, A_3, A_4, B_2, \dots, B_t\}$ becomes a C_5^*-construction for the modified function p', if $t > 1$, and it becomes a C_4^*-construction for the modified function p', if $t = 1$ (the analogous statement holds for an arbitrary $a \in A_i \cap V^+$ and $b \in B_j \cap V^+$, too).*

Proof. Let us first prove $1 = p(A_1 \cup A_2 \cup \bigcup_{j\in J} B_j)$ of (i). Claim 1.8 applied to $A_1 \cup A_2$ and $A_1 \cup B_j$ ($j = 1, 2, \dots, t$) gives that $1 = p(A_1 \cup A_2) \le p(A_1 \cup A_2 \cup \bigcup_{j\in J} B_j) \le p(A_1 \cup A_2 \cup \bigcup_{j=1}^t B_j) = p(A_3 \cup A_4) = 1$ for all $J \subseteq \{1, 2, \dots, t\}$, what was to be proved. The same proof gives that $1 = p(A_1 \cup A_2 \cup A_3 \cup \bigcup_{j\in J} B_j)$ for all $J \subseteq \{1, 2, \dots, t\}$, which is the second statement of (i).

Now we prove (ii). By (i), $p(A_1 \cup A_2 \cup A_3) = p(A_3 \cup A_4 \cup A_1) = 1$. Let us apply $(\cap\cup)$ for these two sets to get that $p(A_1 \cup A_2 \cup A_3 \cup A_4) \ge 2$. Choose an arbitrary $i \in \{1, 2, \dots, t\}$ and apply Claim 1.8 for sets $A_1 \cup A_2 \cup A_3 \cup A_4$ and $A_1 \cup B_j$ ($j \in \{1, 2, \dots, t\} - i$) to get that $2 \le p(A_1 \cup A_2 \cup A_3 \cup A_4 \cup \bigcup_{j\in J} B_j) \le p(A_1 \cup A_2 \cup A_3 \cup A_4 \cup \bigcup_{j\in\{1,2,\dots,t\}-i} B_j) = p(B_i) = 2$ for all $J \subsetneq \{1, 2, \dots, t\}$, proving (ii).

Let us prove (iii): first note that by $(\cap\cup)$ applied to $A_1 \cup \bigcup_{j\in J} B_j$ and $A_3 \cup \bigcup_{j\in J} B_j$ we get that $p(A_1 \cup A_3 \cup \bigcup_{j\in J} B_j) \ge 0$ for any $J \subseteq \{1, \dots, t\}$. Assume that $p(A_1 \cup A_3 \cup \bigcup_{j\in J} B_j) > 0$ for some $J \subseteq \{1, \dots, t\}$. Apply Claim 1.8 to $A_1 \cup A_3 \cup \bigcup_{j\in J} B_j$ and $A_1 \cup B_j$ ($j \in \{1, 2, \dots, t\} - J$) to get that $1 \le p(A_1 \cup A_3 \cup \bigcup_{j\in J} B_j) \le p(A_1 \cup A_3 \cup \bigcup_{j=1}^t B_j) = p(A_2 \cup A_4) \le 0$, a contradiction.

To prove (iv) observe that $m(X \cap Y) \leq m(Y)$ and $m(Y - X) \leq m(Y)$, so the tightness of Y implies that $p(X \cap Y) \leq p(Y)$ and $p(Y - X) \leq p(Y)$. Thus, applying $(\cap\cup)$ for X and Y gives that $p(X) \leq p(X \cup Y)$, and applying $(-)$ for X and Y gives that $p(X) \leq p(X - Y)$. Similarly, $m(X \cap Y) < m(Y)$ implies $p(X) < p(X \cup Y)$, and $m(Y - X) < m(Y)$ implies $p(X) < p(X - Y)$.

Assume that (v) does not hold and choose an X violating (v) with maximum p-value. Let Y be a member of $\mathcal{A}$ crossing X: since $m(Y) > 0$, at least one of $m(X \cap Y) < m(Y)$ and $m(Y - X) < m(Y)$ holds, so (iv) gives that $p(X \cup Y)$ or $p(X - Y)$ is strictly greater than $p(X)$. Let X' be either $X \cup Y$ or $X - Y$ such that $p(X') > p(X)$, so X' cannot cross other sets of the partition $\mathcal{A}$ and $p(X') \geq 3$, contradicting the previous results (note that if $X' \cup Y' = V$ for some $Y' \in \mathcal{A}$ then $p(X') = p(V - X') \leq m(V - X') \leq 2$ also follows).

To prove (vi) assume that a dangerous set X contains the positive nodes $u \in A_1 \cup B_{j_1}$ and $v \in A_3 \cup B_{j_2}$. Note that $p(X) = 2$ can only hold for $X = B_{j_1} \cup B_{j_2}$, which is not dangerous, implying that $p(X) = 1 = m(X) - 1$ must be the case. Furthermore we can assume that $X \cap Y = \emptyset$ for any $Y \in \mathcal{A}$ satisfying $m(Y \cap X) = 0$, since we could substitute X with $X - Y$ by (iv). Similarly, $Y \subseteq X$ can be assumed about any $Y \in \mathcal{A}$ satisfying $m(Y - X) = 0$, since we could substitute X with $X \cup Y$ by (iv). So $u \in A_1$ and $v \in A_3$ would imply that $X = A_1 \cup A_3$, which is not dangerous. On the other hand, if $u \in B_{j_1}$ then $m(B_{j_1} - X) < m(B_{j_1})$ implies by (iv) that $p(X - B_{j_1}) \geq 2$, which cannot be the case since $m(X - B_{j_1}) = 1$.

The proof of (vii) is simple using the properties we have proved so far. □

Claim 5.32. *If $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$ is a C_6^*-construction for p, then $p(X) \leq 1$ for any $X \subseteq V$.*

Proof. Assume that $p(X) > 1$ for some $X \subseteq V$ and assume that we have chosen a maximal X with this property. Note that $X \subseteq A_i$ (or $V - X \subseteq A_i$) cannot hold for some i by Definition 5.28.4. Thus a set A_i cannot cross X since then Claim 1.7 for X and A_i would give that $p(X \cup A_i) \geq p(X)$, contradicting the maximality of X. Thus, by Definition 5.28, X has to be the union of exactly 3 members of $\mathcal{A}$. Apply again Claim 1.7 for X and $A_i \cup A_{i+1}$ with some $A_i \subseteq X$ satisfying $A_{i+1} \subseteq V - X$ to get a contradiction. □

Claim 5.33. *If $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$ is a C_6^*-construction for p and $m \in C(p)$ is minimal then the pair $a_i \in A_i \cap V^+$ and $a_{i+2} \in A_{i+2} \cap V^+$ is admissible for any $i = 1, 2, 3, 4, 5, 6$. Splitting-off such a pair we get C_4^*-construction (for the modified function $p' = p - d_{(V,\{a_i a_{i+2}\})}$).*

Proof. Without loss of generality let $i = 1$. The first statement follows from Claim 5.32: if the pair a_1, a_3 was not admissible then the set blocking X it would have $p(X) = 1$, and

$X = A_1 \cup A_3$ would hold by Claim 5.17, contradicting Definition 5.28.3. For the second statement, observe that $(\cap\cup)$ for $A_1 \cup A_2$ and $A_2 \cup A_3$ gives that $p(A_1 \cup A_2 \cup A_3) = 1$. Using this one can verify that $\{A_1 \cup A_2 \cup A_3, A_4, A_5, A_6\}$ becomes a C_4^*-construction. □

Lemma 5.34. *If $\mathcal{A}$ is a construction for p and $m \in C(p)$ is minimal then there exists a complete admissible splitting-off. Consequently, $dim(p) - 1 \le SLB(p)/2$, if a construction exists for p.*

Proof. Since in a basic C_4^*-construction the splitting-off at nodes $a \in A_1 \cap V^+$ and $b \in A_3 \cap V^+$ is admissible, Claim 5.30, 5.31 and 5.33 give the statement. □

Claim 5.35. *If $\mathcal{A}$ is a construction and $m \in C(p)$ is minimal, then every $X \in \mathcal{A}$ is a maximal tight set.*

Proof. Assume that X' is tight for some $X' \supsetneq X \in \mathcal{A}$. For any $Y \in \mathcal{A}$, if $X' \cap Y \neq \emptyset$ then $Y \subseteq X'$ must hold (since otherwise X' and Y would be properly intersecting, and then by Claim 2.15 they would be crossing, consequently $X' \cup Y$ would also be tight). Thus X' is the union of some members of $\mathcal{A}$.

1. If $\mathcal{A}$ is a C_4^*-construction then $A_i \cup A_{i+1}$ is not tight by Definition 5.26.1, $A_i \cup A_{i+2}$ is not tight by Definition 5.26.3 and $A_i \cup A_{i+1} \cup A_{i+2}$ is not tight by Claim 5.29.

2. If $\mathcal{A}$ is a C_5^*-construction then the union of some members of $\mathcal{A}$ cannot be tight by Claim 5.31.

3. If $\mathcal{A}$ is a C_6^*-construction then the union of some members of $\mathcal{A}$ cannot be tight by Claim 5.32.

□

The following lemma shows that if a construction exists then it is unique.

Lemma 5.36. *If $\mathcal{A}$ is a partition of V then it satisfies at most one of Definition 5.26, Definition 5.27, and Definition 5.28. If $\mathcal{A}$ and $\mathcal{A}'$ are two different partitions of V then at most one of them is a construction.*

Proof. The first statement is straightforward: the three definitions require different number of partition classes or different function values of the classes. The second statement is due to the fact that choosing an arbitrary minimal $m \in C(p)$, every set of a construction is a maximal tight set. □

5.4.2 The degree-specified problem

Let us be given a symmetric, positively crossing supermodular function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ with a maximizing oracle, a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_r\}$ of the set V and a degree-specification $m \in \mathbb{Z}_+^V$. In this section we want to solve the degree-specified version of Problem 5.25.

In Section 5.4.2 we will assume that the classes of $\mathcal{P}$ are indexed such that

$$m(P_1) \geq m(P_i) \text{ for any } i = 1, 2, \ldots, r. \tag{5.13}$$

We will also say that the colour of P_1 is **red**, and any other colour is **non-red**. Note that under the assumption (5.13) the condition (5.11) is equivalent to requiring that $m(P_1) \leq m(V - P_1)$.

Before giving the description of the obstacles we first describe the building blocks of our algorithm for solving the problem at hand. The algorithm will be similar to Algorithm SYMCROS_COVER: we perform splitting-off, one-change and edge-switch operations while only taking care of the allowedness of the degree specification. Therefore we introduce the following operations.

Assume that we are given an allowed degree-specification m (we emphasize that this does not require the dimension condition (5.12)). An admissible splitting-off at u and v is called **allowed** if $c(u) \neq c(v)$ and the degree-specification after the splitting-off is again allowed. This last condition is equivalent to requiring that if $m(P_1) = m(V)/2$ then an allowed splitting-off uses exactly one red node. After an allowed splitting-off we will always reindex the classes of the partition $\mathcal{P}$ in order to maintain that $m(P_1)$ is maximal. A **complete allowed splitting-off** is a sequence of allowed splitting-off steps that decreases $m(V)$ to zero. Observe that a solution to the degree-specified problem exists if and only if there is a complete allowed splitting-off.

Let $uv \in E(G)$ be a split edge and $x \in V^+$ and consider the edge-switch operation at x, u, v. If m was allowed, $c(u) \neq c(x)$, and m' is allowed, then the edge-switch operation is **allowed**. Pick an edge $uv = e$ of G and (distinct) positive nodes x and y. The admissible one-change at x, u, v, y is an **allowed one-change (at x, u, v, y)** if $c(u) \neq c(x)$, $c(v) \neq c(y)$ and the degree-specification after the one-change is still allowed.

After performing any of the above operations we will always replace the functions m and p with the modified functions m' and p'. We will also reindex the classes of the partition $\mathcal{P}$ in order to maintain that $m(P_1)$ is maximal.

Obstacles

In this section we give the description of the instances of the degree-specified Problem 5.25 that satisfy the necessary conditions (5.9)-(5.11), but still not solvable. It will be useful to give these definitions again in two levels, therefore first we give some preliminary definitions. Note that in the following definitions only the allowedness of the degree-specification m is assumed, the dimension condition (5.12) will be a consequence. The following definitions assume that we are given an instance of the degree-specified version of Problem 5.25.

Definition 5.37. *A C_4^*-construction $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ for p is called a **C_4^*-semiobstacle for $(p, \mathcal{P}, m)$** if (5.9)-(5.11) holds and $m(V) = SLB(p)$.*

Definition 5.38. *A C_5^*-construction $\mathcal{A} = \{A_1, A_2, A_3, A_4, B_1, \ldots, B_t\}$ for p is called a **C_5^*-semiobstacle for $(p, \mathcal{P}, m)$** if (5.9)-(5.11) holds, and $m(V) = SLB(p)$.*

Definition 5.39. *A C_6^*-construction $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$ for p is called a **C_6^*-semiobstacle for $(p, \mathcal{P}, m)$** if (5.9)-(5.11) holds, and $m(V) = SLB(p)$.*

In other words, a semiobstacle is nothing else but a construction $\mathcal{A}$ and an allowed degree-specification m achieving the lower bound $m(V) = SLB(p)$. Note that such a degree-specification does not necessarily exist (i.e. the existence of a construction does not imply the existence of a semiobstacle).

Definition 5.40. *A C_4^*-semiobstacle $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ for $(p, \mathcal{P}, m)$ is called a **C_4^*-obstacle for $(p, \mathcal{P}, m)$** if there exists an $i_0 \in \{1, 2\}$ such that the positive nodes of $A_{i_0} \cup A_{i_0+2}$ all have colour 1.*

Observe that $m(P_1) = m(V - P_1) = m(V)/2$ in a C_4^*-obstacle. A C_4^*-obstacle will be called **basic**, if $m(V) = 4$.

Definition 5.41. *A C_5^*-semiobstacle $\mathcal{A} = \{A_1, A_2, A_3, A_4, B_1, \ldots, B_t\}$ for $(p, \mathcal{P}, m)$ is called a **C_5^*-obstacle for $(p, \mathcal{P}, m)$** if*

1. *Either $m(P_1) = m(V - P_1)$ and there is an $i_0 \in \{1, 2\}$ such that $m(P_1 \cap A_{i_0}) = m(P_1 \cap A_{i_0+2}) = m(P_1 \cap B_j) = 1$ for every $j = 1, 2, \ldots, t$,*
2. *Or $m(P_1) = m(P_2) = m(V)/2 - 1$, $m(P_1 \cap A_1) = m(P_1 \cap A_3) = m(P_2 \cap A_2) = m(P_2 \cap A_4) = 1$, and there exists a $j_0 \in \{1, 2, \ldots, t\}$ such that $m(P_1 \cap B_j) = m(P_2 \cap B_j) = 1$ for any $j \in \{1, 2, \ldots, t\} - j_0$, (consequently $m((P_1 \cup P_2) \cap B_{j_0}) = 0$).*

Definition 5.42. *A C_6^*-semiobstacle $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$ for $(p, \mathcal{P}, m)$ is called a **C_6^*-obstacle for $(p, \mathcal{P}, m)$** if the positive node in A_i and in A_{i+3} have the same colour for every $i = 1, 2, 3$.*

Observe that the C_6^*-obstacle is quite restricted: it only exists with $m(V) = 6$ and if there are at least 3 colours (i.e. $|\mathcal{P}| \geq 3$). This last statement is because the positive nodes in A_1, A_2 and A_3 must have different colours. Note that the definition of the obstacles (and that of semiobstacles) also allows cyclical reindexing and reversing the order of the sets A_i.

Lemma 5.43. *If a C_5^*-semiobstacle $\mathcal{A} = \{A_1, A_2, A_3, A_4, B_1, \ldots, B_t\}$ is not a C_5^*-obstacle, then there exists a complete allowed splitting-off.*

Proof. Assume that $\mathcal{A}$ is a semiobstacle for $(p, \mathcal{P}, m)$ that is not a C_5^*-obstacle. Let $\{a_i\} = A_i \cap V^+$ for every $i = 1, 2, 3, 4$ and $\{b_j, b'_j\} = B_j \cap V^+$ for every $j = 1, 2, \ldots, t$ (where $b_j = b'_j$ might even hold for some values of j, and $V^+ = \{v \in V : m(v) > 0\}$). We use the observation that if $t > 1$ then the admissible splitting-off at $a \in A_i$ and $b \in B_j$ for some $i \in \{1, 2, 3, 4\}$ and $j \in \{1, 2, \ldots, t\}$ gives a new C_5^*-semiobstacle $\mathcal{A} - \{A_i, B_j\} + \{A_i \cup B_j\}$ for the modified p and m. Similarly, if $t = 1$ then this way we obtain a basic C_4^*-semiobstacle.

Our approach is the following: we perform an arbitrary allowed splitting-off at some $a \in A_i$ and $b \in B_j$ and we check that the semiobstacle after this step is an obstacle or not. If it is then we show how to modify the step in order to avoid the trouble: in most of the cases this modification will be an allowed edge-switch. This way we get an inductive proof on t and it is easy to see that the $t = 0$ base case is true: there is a complete allowed splitting-off in a basic C_4^*-semiobstacle which is not a C_4^*-obstacle.

Assume that after the allowed splitting-off at a_1 and b_1, say, the partition $\mathcal{A}' = \{A_1 \cup B_1, A_2, A_3, A_4, B_2, \ldots, B_t\}$ is a (C_5^*- or a basic C_4^*-) obstacle for $(p', \mathcal{P}, m')$, where p' and m' are the modified functions. In this proof we will assume that $m'(P_1) \geq m'(P_2) \geq \cdots \geq m'(P_r)$. As usual, we will say that the colour of P_1 is red. Distinguish the following cases.

1. Definition 5.41.1 (first option of a C_5^*-obstacle) or Definition 5.40 (basic C_4^*-obstacle) holds for $\mathcal{A}'$ (with p', m' of course). Assume that the colour of b_j is red for every $j = 2, \ldots, t$. There are two subcases.

   (a) a_2, a_4 are both red. Then b'_1 is not red (since the splitting we have performed was allowed) and b_1 is not red either (otherwise $\mathcal{A}$ would have been a C_5^*-obstacle for $(p, \mathcal{P}, m)$ satisfying Definition 5.41.1). Perform the edge-switch at a_2, b_1, a_1, i.e. replace the edge $a_1 b_1$ with the edge $a_2 b_1$. Let the modified functions be m'' and p'': note that $m''(V) = m'(V)$ and $m''(P_1) \leq m'(P_1)$. This edge-switch is allowed, since $c(a_2) \neq c(b_1)$ as we have seen, $m''(P_1) \leq m'(P_1)$ and $c(a_1) = c(a_3) = c(b'_1) = c(b'_2) = \cdots = c(b'_t)$ cannot hold, since then $\mathcal{A}$ would be a C_5^*-obstacle for $(p, \mathcal{P}, m)$ satisfying Definition 5.41.1 with colour class P_2. Moreover $\mathcal{A}'' = \{A_1, A_2 \cup B_1, A_3, A_4, B_2, \ldots, B_t\}$ cannot be an obstacle for

$(p'', \mathcal{P}, m'')$, since that would mean that there is a colour class $P \in \mathcal{P} - P_1$ such that $a_1, a_3 \in P$ and $b'_j \in P$ for every $j = 2, \ldots, t$ (note that Definition 5.41.2 cannot hold for $\mathcal{A}''$ since $c(b'_1) \neq c(a_4)$). But this implies that $\mathcal{A}$ was a C_5^*-obstacle for $(p, \mathcal{P}, m)$: if $m(P \cap B_1) = 1$ then it satisfies Definition 5.41.1 with class P and if $m(P \cap B_1) = 0$ then it satisfies Definition 5.41.2 with classes P_1 and P ($m(P \cap B_1) = 2$ cannot be the case since then $m(P) > m(V)/2$).

(b) b'_1, a_3 are both red. Here we also assume that $m'(P_2) < m'(V)/2$, since the $m'(P_1) = m'(P_2) = m'(V)/2$ case was handled just before (switch P_1 and P_2). Since $c(a_1) \neq c(b_1)$, at most one of a_1 and b_1 is red. Moreover a_1 cannot be red, since then $\mathcal{A}$ would have been a C_5^*-obstacle for $(p, \mathcal{P}, m)$ satisfying Definition 5.41.1 with class P_1. If b_1 is red then replace the edge $a_1 b_1$ with the edge $a_2 b'_1$: since the number of red positive nodes does not change, this way we destroyed the obstacle, since $c(b'_1) \neq c(a_4)$. If b_1 is not red, either, then replace the edge $a_1 b_1$ with the edge $a_1 b'_1$ (i.e. apply the edge-switch at b'_1, a_1, b_1). Let the modified functions be m'' and p''. Since $c(a_2) = c(a_4) = c(b'_2) = \cdots = c(b'_t)$ cannot hold by $m'(P_2) < m'(V)/2$, this edge-switch is allowed. For the same reason, this way we get rid of the obstacle: neither Definition 5.41.1 nor Definition 5.40 can hold for $\mathcal{A}'$ and $p'', \mathcal{P}, m''$ since $c(a_2) = c(a_4) = c(b'_2) = \cdots = c(b'_t)$ is not true, and Definition 5.41.2 cannot hold since $c(b_1) \neq c(a_3)$.

2. Definition 5.41.2 holds for $\mathcal{A}'$ and $p', \mathcal{P}, m'$. Let P_1 be the colour of b'_1 and a_3, and let P_2 be the colour of a_2 and a_4. Replace the edge $a_1 b_1$ with the edge $a_2 b'_1$ (note that this is not an edge-switch). Let the modified functions be m'' and p''. Since $m''(P_1) \leq m'(P_1)$, $m''(P_2) \leq m'(P_2)$ and $c(a_1) \neq c(b_1)$, we have $m''(P_i) \leq m''(V)/2$ for every $i = 1, 2, \ldots, r$ (that is, m'' is allowed). Furthermore, Definition 5.41.2 cannot hold any more, since $c(b_1) = c(a_4)$ and $c(a_1) = c(a_3)$ would imply that $\mathcal{A}$ was a C_5^*-obstacle for $(p, \mathcal{P}, m)$ satisfying Definition 5.41.2 with colour classes P_1, P_2.

□

The following result motivates the definition of the obstacles.

Lemma 5.44. *If there is an obstacle $\mathcal{A}$ for $(p, \mathcal{P}, m)$, then there is no complete allowed splitting-off (though there is a complete admissible splitting off).*

Proof. In this proof, a_i will denote a node of $A_i \cap V^+$ and b_i one of $B_i \cap V^+$. We first prove the result when there exists a C_4^*-obstacle. The two other cases reduce to it.

1. If $\mathcal{A}$ is a C_4^*-obstacle, we may assume that a maximum colour class is incident to $A_1 \cup A_3$. Each allowed splitting-off has to contain one positive node of $A_1 \cup A_3$ and

one of $A_2 \cup A_4$, therefore $m(A_i \cup A_{i+1}) - p(A_i \cup A_{i+1})$ may only decrease with an even number after any sequence of allowed splitting-off. Then Definition 5.26.1 and Definition 5.26.2 imply that $m(A_i)$ may never decrease to zero for any i. Therefore there is no complete allowed splitting-off.

2. If $\mathcal{A}$ is a C_5^*-obstacle, then there are two cases. Since every set of $\{B_1, \dots B_t\}$ is tight, an allowed splitting-off may not contain two positive nodes of the same set of $\{B_1, \dots B_t\}$.

   (a) Definition 5.41.1 holds. Since the splitting-off at a_i, a_{i+1} is not admissible for any positive nodes $(a_i, a_{i+1}) \in A_i \times A_{i+1}$, each allowed splitting-off involves a positive node of a set B_i. Perform an allowed splitting-off (either at a_i, b_j or at $b_i, b_j, j \neq i$). Then we get a C_5^*-obstacle, either $\mathcal{A} \cup (A_i \cup B_j) - A_i - B_j$ or $\mathcal{A} \cup (B_i \cup B_j) - B_i - B_j$. Note that the number of sets of the obstacle decreased by one. Repeat until the obstacle consists of four sets, then by Claim 5.31 it is a C_4^*-obstacle and we may apply 1.

   (b) Definition 5.41.2 holds. If there exist a complete allowed splitting-off, then one of its allowed splitting-offs is at u, v with $u \in B_{j_0}$. Perform this splitting-off, then we get a C_5^*-obstacle where 5.41.1 holds, a contradiction.

3. If $\mathcal{A}$ is a C_6^*-obstacle, since $A_i \cup A_{i+1}$ is dangerous and $c(a_i) = c(a_{i+3})$, the only allowed splitting-off are at a_i, a_{i+2}, for $i = 1, 2, 3$. By $(\cap\cup)$ applied to $A_i \cup A_{i+1}$ and $A_{i+1} \cup A_{i+2}$, we have $p(A_i \cup A_{i+1} \cup A_{i+2}) = m(A_i \cup A_{i+1} \cup A_{i+2}) + 2$. Perform an allowed splitting off at a_i, a_{i+2} for some $i \in \{1, 2, 3\}$, then $A_i \cup A_{i+1} \cup A_{i+2}$ is tight and we are back to Case 1 with the C_4^*-obstacle $\{A_i \cup A_{i+1} \cup A_{i+2}, A_{i+3}, A_{i+4}, A_{i+5}\}$.

$\square$

Note that if $\mathcal{A}$ is an obstacle then $\phi = \alpha_p = \sum_{A \in \mathcal{A}} p(A)$.

The splitting-off theorem

Theorem 5.45. *Let $p_0 : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric, positively crossing supermodular set function, $\mathcal{P}$ a partition of V and $m_0 \in \mathbb{Z}_+^V$ an allowed degree-specification so that $\frac{1}{2} m_0(V) \geq \dim(p_0) - 1$. Then there exists a solution of Problem 5.25 satisfying the degree-specification m_0, unless there exists a C_4^*-, C_5^*-, or a C_6^*-obstacle for $(p_0, \mathcal{P}, m_0)$.*

Proof. In the proof of this theorem we will give an algorithm.

First Step of the Algorithm: Perform an arbitrary sequence of allowed splitting-off operations as long as there exists one.

Let G be the graph of the edges split so far, $p = p_0 - d_G$ and $m(v) = m_0(v) - d_G(v)$ for all $v \in V$. Let us characterize the situation when there is no further allowed splitting-off. For symmetric, positively crossing supermodular functions we can extend Lemma 4.6 even for this partition constrained case.

Lemma 5.46. *Let $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ be a symmetric, positively crossing supermodular function and m be an allowed degree specification. If $p(X) > 1$ for some $X \subseteq V$ then there is an allowed splitting-off.*

Proof. Assume that there is no allowed splitting-off and let $M_p = \max\{p(X) : X \subseteq V\}$, which is by assumption at least 2. Let Y be a minimal set satisfying $p(Y) = M_p$. By symmetry, $p(V - Y) = M_p$, too, so we can choose a minimal set $Z \subseteq V - Y$ satisfying $p(Z) = M_p$. We know from the proof of Lemma 4.6 that any pair of positive nodes $y \in Y, z \in Z$ is admissible, so there is an allowed splitting off if we can choose such a red-non-red pair $y \in Y, z \in Z$. Assume that either all positive nodes in $Y \cup Z$ are of colour 1, or none of them are of colour 1. Choose $x \in V^+ - (Y \cup Z)$ of colour different from 1 in the first case, and of colour 1 in the second case, and let $y \in Y$ and $z \in Z$ be arbitrary positive nodes. If the splitting at x and y is not admissible then there is a dangerous set X containing x and y. Since $m(X - Y) \le m(X) - m(y) \le m(X) - 1$ and $p(Y - X) < M_p$ by the minimality of Y, X and Y cannot satisfy $(-)$, since that would mean $m(X) - 1 + M_p \le p(X) + p(Y) \le p(X - Y) + p(Y - X) < m(X - Y) + M_p \le m(X) - 1 + M_p$, a contradiction. So $Y \subseteq X$ must hold. But then X can only be dangerous if $p(X) = M_p$ and $m(X) = m(Y) + 1 = M_p + 1$. Similarly, if the splitting at x and z is not admissible then there exists an X' containing x and Z and satisfying $p(X') = M_p$ and $m(X') = m(Z) + 1 = M_p + 1$. Since X and X' must cross (because even $V^+ - (X \cup X')$ is not empty), this implies that $p(X \cap X') = M_p$, but this contradicts the fact that $m(X \cap X') = m(x) = 1$. ∎

By Lemma 5.46, since any pair $v \in V^+ \cap P_1$ and $u \in V^+ - P_1$ is in a dangerous set X, we have $1 \ge p(X) \ge m(X) - 1 \ge m(\{u, v\}) - 1 \ge 2 - 1$ thus $m \le 1$. Most of the statements of Lemma 5.20 hold here, too, the proofs are only a little more complicated (they are presented here for the reader's convenience).

Lemma 5.47. *If there is no allowed splitting-off, let $V^+ = \{v_1, v_2, \ldots, v_k\}$. Then the following hold.*

(i) *for all $i \in \{1, 2, \ldots, k\}$ there exists a unique maximal tight set V_i containing v_i,*

(ii) *if the splitting at v_i and v_j is not admissible for some $i, j \in \{1, 2, \ldots, k\}$ then the set blocking v_i and v_j is $V_i \cup V_j$,*

(iii) *the sets $V_1, V_2, \ldots, V_k$ form a partition of V.*

Proof. The sets that we consider will always have positive p value, so we can use ($\cap\cup$) and ($-$) if two of them cross. Observe that since $m(V) \geq 4$, a set X blocking a pair $v \in V^+ \cap P_1, u \in V^+ - P_1$ and another set Y blocking a pair $v \in V^+ \cap P_1, w \in V^+ - P_1$ cross each other, implying that $p(X \cap Y) = p(X - Y) = 1$, so every $x \in V^+$ is in a tight set. Similarly, if T_1 and T_2 are two tight sets containing the positive node $x \in V^+$ then T_1 and T_2 cannot cross each other, since then ($-$) would imply that $p(T_1 - T_2) = 1 > m(T_1 - T_2) = 0$, a contradiction. Thus one of T_1 and T_2 must contain the other, so indeed there exists a unique maximal tight set V_i containing v_i for every i.

Let i, j be two different indices between 1 and k such that the splitting-off at v_i and v_j is not admissible. It is straightforward that V_i and V_j have to be disjoint by Claim 5.17. Similarly, a set X blocking v_i and v_j must contain V_i (and V_j) by Claim 5.17. On the other hand, if $l \in \{1, 2, \ldots, k\}$ is different from i and j for which v_i and v_l is not admissible (note that such an l exists), and Y is a set blocking v_i and v_l then ($\cap\cup$) implies that $X \cap Y = V_i$ (since it is tight) and ($-$) implies that $X - Y = V_j$ (since it is tight again). This finishes the proof of (ii).

The only thing to be proved to get (iv) is that $\cup_{i=1}^k V_i = V$: but if this was not the case then Claim 1.9 would also imply that $p(\cup_{i=1}^k V_i) = 1$, which would give a contradiction, since $m(V - \cup_{i=1}^k V_i) = 0$ and $p(V - \cup_{i=1}^k V_i) = p(\cup_{i=1}^k V_i) = 1$. ∎

If $m(V) = 4$ then there may exist an admissible but not allowed splitting-off. However this cannot happen if $m(V) \geq 6$.

Lemma 5.48. *If $m(V) \geq 6$ and there is no allowed splitting-off, then there is no admissible splitting-off, either.*

Proof. Let us introduce the **nonadmissibility graph** N on node set V^+ that contains an edge between two nodes $x, y \in V^+$ (sharing possibly the same colour) if and only if the splitting-off at x and y is not admissible. By the arguments above, if there is no allowed splitting-off then any pair $v \in V^+ \cap P_1, u \in V^+ - P_1$ is connected with an edge in N. Note that for any nonempty subset $Z \subsetneq V^+$ satisfying that $N[Z]$ is connected we have that $p(\cup_{v_i \in Z} V_i) = 1$ by Claim 1.9. This implies that if $v_i, v_j \in V^+$ is any pair and $N - \{v_i, v_j\}$ is connected, then $v_i v_j$ is also an edge of N, since $p(V_i \cup V_j) = 1$ by the symmetry of p.

We have to prove that N is a clique in this case. Let us first consider a pair $x, y \in V^+ - P_1$: since $V^+ - P_1 - \{x, y\}$ is not empty by our assumptions (since m is allowed and $m(V) \geq 6$), it is easy to check that $N - \{x, y\}$ is connected, so $xy \in E(N)$. Finally, if $x, y \in V^+ \cap P_1$ then by the preceding argument $N - \{x, y\}$ is also connected, so again $xy \in E(N)$. ■

Second Step of the Algorithm, First Main Case: If there does not exist an admissible splitting-off, then perform an arbitrary sequence of allowed one-change operations, until $m(V)$ decreases to 2, when the procedure can be finished with a last allowed splitting-off. Output the graph found and terminate.

Let us prove the correctness of this part of the algorithm. We can use the observations made after the First Step of Algorithm SYMCROS_COVER. One such observation is that there are no split edges between the classes of our tight partition. Also, by Lemma 5.20 (iii), an allowed one-change operation will not create an admissible splitting-off, unless $m(V)$ becomes 2. The key observation is given in the following Lemma.

Lemma 5.49. *Assume that there is no admissible splitting-off and there is no allowed one-change. Then there does not exist an admissible one-change, either.*

Proof. Let $uv \in E(G)$: since there is no admissible splitting-off, $uv \subseteq V_i$ for some i. We will show that there is no admissible one-change using this edge. Let $v_r \in V^+ \cap P_1$, $v_s \in V^+ - P_1$ be different from v_i. By possibly exchanging u and v we can achieve that $u \notin P_1$ and the colour of v and v_s is different. Since the one-change at v_r, u, v, v_s is not allowed, there must exist a switchblocking set X, without loss of generality we assume that it is (the unique) (v_r, u, v)-switchblocking set. By Claim 5.21 there is no admissible one-change at this edge, since $p(V_j \cup (X \cap V_i)) = 0$ for all $j \neq i$, what was to be proved. ■

This lemma shows that we can again refer to the arguments given in the previous section, in particular we can thus prove the correctness of this case of the algorithm. We only need to prove that this case of the algorithm cannot get stuck, we will always find an allowed one-change, unless $m(V)$ decreases to 2. But if this was not the case, then the steps that we have done so far can be considered as a running of the Algorithm SYMCROS_COVER, and by the previous lemma the Algorithm SYMCROS_COVER would get stuck after these steps, too. But the Algorithm SYMCROS_COVER cannot get stuck with such an input as ours, a contradiction.

Now we can continue with the description of the Second Main Case of the second step of the algorithm.

Second Step of the Algorithm, Second Main Case: If there exists an admissible splitting-off (in which case $m(V) = 4$ by Lemma 5.48), then try to find an allowed one-change. If this is found then finish the procedure with a last allowed splitting-off, output the graph found and terminate.

Before continuing the description of the algorithm first we prove some lemmas about the case when there is no allowed one-change. Let the graph of the edges split so far be denoted by G, $p = p_0 - d_G$ and $m(v) = m_0(v) - d_G(v)$ for every $v \in V$. Let $V^+ = \{v_1, v_2, v_3, v_4\}$ indexed in the way such that the splitting-off at v_1 and v_3 is admissible (and then of course the one at v_2, v_4 is admissible, too, but either $c(v_1) = c(v_3)$, or $c(v_2) = c(v_4)$), and let V_i be the maximal tight set containing v_i for all i. Since the splitting-off at v_1 and v_3 is admissible (but not allowed), G might contain some edges between the classes V_i, but only between consecutive ones (i.e. between V_i and V_{i+1} for some $i \in \{1, 2, 3, 4\}$).

By Lemma 5.19 (2d) the obstacle of the admissibility of the one-change along an edge $uv \in E(G)$ **induced** in one of the sets V_i and two positive nodes distinct from v_i is a switchblocking set. Furthermore, we can show the following.

Lemma 5.50. *Assume that $e = uv$ is an edge of G induced in V_1, say. Then there is no (v_3, e)-switchblocking set, consequently either the one-change at v_3, u, v, v_2, or the one at v_3, v, u, v_2 is admissible. (The same is true with v_4 instead of v_2 here!)*

Proof. Assume indirectly that X is such a switchblocking set: but since X and V_1 cross each other this implies that $p(X \cup V_1) = 1$, contradicting the admissibility of the splitting-off at v_1 and v_3. ■

Lemma 5.51. *Assume that uv is an edge of G such that $u \in V_1$ and $v \in V_2$. Then either $c(u) = c(v_1) = c(v_3)$ or $c(v) = c(v_2) = c(v_4)$.*

Proof. Otherwise there would be an allowed one-change. Without loss of generality we can assume that $c(v_1) = c(v_3) = 1$. Assume that $u \notin P_1$. If $c(v) \neq c(v_4)$ then the one-change at v_3, u, v, v_4 would be allowed, so $c(v) = c(v_4)$. If $c(v_2) \neq c(v_4)$ then the one-change at v_3, u, v, v_1 is allowed by Lemma 5.19 (3c). ■

The next lemma is about a case when there is an edge induced in V_1.

Lemma 5.52. *Assume that the one-change at v_2, u, v, v_4 is admissible where the edge uv is induced in V_1. Then there exists a (v_2, v, u)-switchblocking set X_2, and a (v_4, u, v)-switchblocking set X_4. These sets are disjoint, $p(V_1 - X_2) = p(V_1 - X_4) = 0$ and $p(V_1 - (X_2 \cup X_4)) = 1$. Furthermore $c(v_2) = c(u)$ and $c(v) = c(v_4)$ both must hold, consequently the colour of v_2 and that of v_4 must be different.*

Proof. The existence of X_2 follows from Lemma 5.50: if it does not exist then both the one-changes at v_3, u, v, v_2 and at v_3, v, u, v_2 are admissible and one of them would be allowed. Similarly for X_4. If $X_2 \cap X_4 \neq \emptyset$ then (1.5) for these two sets give that $p(X_2 \cup X_4) = 1$, which cannot be the case by Claim 5.17. Let $Y_i = V_1 - X_i$ for both $i = 2, 4$: $(-)$ for V_1 and X_i gives that $p(Y_i) \geq 0$ for both $i = 2, 4$. Now (1.5) for Y_2 and Y_4 gives that $p(Y_2) = p(Y_4) = 0$ and $p(Y_2 \cap Y_4) = 1$, as claimed, since $Y_2 \cap Y_4 = V_1 - (X_2 \cup X_4)$.

To prove the last statement assume that for example $c(v_2) \neq c(u)$: then the one-change at v_2, u, v, v_3 would be allowed, unless $c(v) = c(v_3)$, but in this case the one-change at v_3, u, v, v_4 would be allowed, a contradiction. ■

End of the Algorithm: Let V_1, V_2, V_3, V_4 be the partition into maximal tight sets and $v_i \in V_i$ be the positive nodes, s.t. v_1 and v_3 is admissible (note that we continue the description of the Second Step, Second Main Case).

CASE A There is an edge $e \in E(G)$ induced in a class V_i and another edge $f \in E(G)$ between consecutive classes: find a complete allowed splitting-off by unsplitting these two edges.

CASE B Every edge of G goes between consecutive classes V_j and V_{j+1} (including the case $E(G) = \emptyset$): if either $c(v_1) = c(v_3) = c(e \cap (V_1 \cup V_3))$ for every edge $e \in G$, or $c(v_2) = c(v_4) = c(e \cap (V_2 \cup V_4))$ for every edge $e \in G$, then $\{V_1, V_2, V_3, V_4\}$ was a C_4^*-obstacle, otherwise a complete allowed splitting-off can be found (by unsplitting only 2 edges of G).

CASE C Every edge of G is induced in a class of the partition $\{V_1, V_2, V_3, V_4\}$.

Subcase (i) There is an edge uv induced in V_1, say, for which the one-change at v_2, u, v, v_4 is admissible. If uv is the only edge of G then there was a C_6^*-obstacle, otherwise a complete allowed splitting-off can be found (by unsplitting 2 edges).

Subcase (ii) For any edge uv of G induced in some V_i, the one-change neither at v_{i-1}, u, v, v_{i+1} nor at v_{i-1}, v, u, v_{i+1} is admissible. Then there exists a partition satisfying C_5^*-semiobstacle in the input: if it is not a C_5^*-obstacle (which is easy to check) then find a complete allowed splitting-off.

Let us first justify **CASE A**.

Lemma 5.53. *If G contains an edge induced in some set V_i, and another edge between some V_j and V_{j+1} then we can find a complete allowed splitting-off after unsplitting these two edges.*

Proof. Assume that $e = uv$ is an edge of G induced in V_1, say, and G also contains an edge f between two classes, too. We first prove the following claim.

Claim 5.54. *It is not possible that (v_i, u, v)-switchblocking sets exist for both $i = 2$ and $i = 4$. (Consequently, either the one-change at v_2, u, v, v_4, or the one at v_2, v, u, v_4 is admissible.)*

Proof. Assume indirectly that X_i is a (v_i, u, v)-switchblocking set for both $i = 2$ and 4 and consider two cases. In both cases we can assume that one endpoint of f is in V_2.

In the first case the edge f is between V_2 and V_3: apply $(\cap\cup)$ for X_2 and X_4 to get that $p(X_2 \cup X_4) = 0$, then apply $(\cap\cup)$ for $X_2 \cup X_4$ and $V_3 \cup V_4$: using the edge f you get a contradiction. In the other case the edge f is between V_1 and V_2: apply $(\cap\cup)$ for X_2 and V_1 to get that f must be induced in X_2 and then apply $(-)$ for X_4 and V_1 to see that f cannot enter X_4. This contradicts Lemma 5.19 (2b).

By possibly exchanging the role of u and v we can assume that the one-change at v_2, u, v, v_4 is admissible. Let $f = xy$ such that $y \in V_2$: by Lemma 5.51 and Lemma 5.52, x must be red in this case. We have again two cases: either $x \in V_1$ or $x \in V_3$. In both cases $G - e - f + ux + vv_3 + v_2v_4 + v_1y$ is a graph that satisfies our requirements. This can be justified the following way. By Lemma 5.52, there exists a (v_2, v, u)-switchblocking set X_2, and a (v_4, u, v)-switchblocking set X_4. Applying $(\cap\cup)$ for X_2 and $V_2 \cup V_3$ gives that $p(V_3 \cup X_2) = 0$. Similarly, $(\cap\cup)$ for X_4 and V_1 gives that $p(X_4 \cap V_1) = 0$. Now apply the edge-switch operation at v_3, v, u, which is clearly allowed by Lemma 5.50 and let p' and m' be the modified functions. By the preceding observations, the partition $V_1 - (X_2 \cup X_4), X_4 \cap V_1, V_3 \cup X_2, V_4$ is a p'-tight partition (clearly $p' \le 1$). Now we claim that the one-change at v_1, y, x, u became admissible. By Lemma 5.19 (2d) we only have to show that there is no (v_1, y, x)-switchblocking set, and there is no (u, x, y)-switchblocking set, either (with respect to p' and m', of course). If X was a (v_1, y, x)-switchblocking set then (1.6) for X and V_2 and p' would give a contradiction (note that $p'(V_2) = p(V_2) = 1$). On the other hand if X was a (u, x, y)-switchblocking set then $(\cap\cup)$ for X and $V_3 \cup X_2$ would give that $p'(V_3 \cup X_2 \cup X) = 1 = p(V_3 \cup X_2 \cup X)$, contradicting Claim 5.17 (applied to p), since this set crosses V_1. Figure 5.4.2 illustrates the proof. ∎

We are left with two cases: either all edges of G go between (consecutive) classes of V_1, V_2, V_3, V_4, or all of them are induced in the classes of V_1, V_2, V_3, V_4. Let us first describe **CASE B**, when **every edge of G goes between (consecutive) classes** of the partition V_1, V_2, V_3, V_4.

Lemma 5.55. *Assume that G contains only edges between the classes V_i. If either $c(v_1) = c(v_3) = c(e \cap (V_1 \cup V_3))$ for every edge $e \in G$, or $c(v_2) = c(v_4) = c(e \cap (V_2 \cup V_4))$ for every edge $e \in G$ then $\{V_1, V_2, V_3, V_4\}$ was a C_4^*-obstacle, otherwise a complete allowed splitting-off can be found by unsplitting two edges of G.*

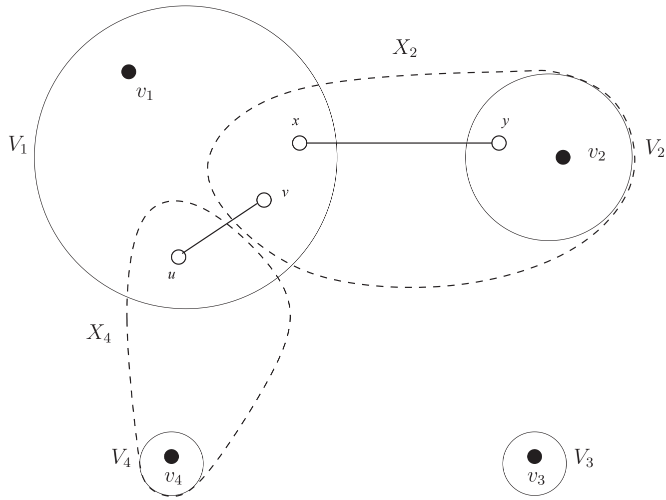

Figure 5.2: Illustration for the proof of Lemma 5.53.

Proof. Assume that there is an edge e such that $c(v_1) \neq c(e \cap (V_1 \cup V_3))$ and an edge f such that $c(v_2) \neq c(f \cap (V_2 \cup V_4))$. These imply by Lemma 5.51 that $c(v_2) = c(v_4) = c(e \cap (V_2 \cup V_4))$ and $c(v_1) = c(v_3) = c(f \cap (V_1 \cup V_3))$. Assume without loss of generality that $e = uv$ where $u \in V_1$ and $v \in V_2$.

Apply the edge-switch operation at v_1, v, u which is allowed by Lemma 5.19 (3a) and note that the function value of the sets V_i and $V_i \cup V_{i+1}$ $(i = 1, 2, 3, 4)$ does not change after this edge-switch. Furthermore, since the splitting at v_3 and u is admissible after this edge-switch by Lemma 5.19 (3c) (since the one-change at v_3, u, v, v_1 was admissible originally), this means that we are again at the Second Main Case of our algorithm (with the same tight partition V_1, V_2, V_3, V_4), and by Lemma 5.51 we created an allowed one-change using edge f. ∎

Now we can describe **CASE C**, when **every edge of G is induced in the classes** of the partition V_1, V_2, V_3, V_4. **The first subcase** of this case is when there is an edge uv induced in V_1, say, for which the one-change at v_2, u, v, v_4 is admissible. The next lemma deals with this case.

Lemma 5.56. *Assume that G contains and edge uv induced in V_1 and the one-change at v_2, u, v, v_4 is admissible. If uv is the only edge of G then there was a C_6^*-obstacle, otherwise* *a complete allowed splitting-off can be found by unsplitting uv and an arbitrary other edge of G.*

Proof. By Lemma 5.52 there exists a (v_2, v, u)-switchblocking set X_2, and a (v_4, u, v)-switchblocking set X_4. Let $Y_i = V_1 - X_i$ for $i = 2, 4$ and let $A_1 = Y_2 \cap Y_4$, $A_2 = Y_4 - Y_2 = V_1 \cap X_2$, $A_3 = V_2$, $A_4 = V_3$, $A_5 = V_4$ and $A_6 = Y_2 - Y_4 = V_1 \cap X_4$. By Lemma 5.52 $p(A_1) = 1$, and similarly $p(A_2) = p(A_6) = 0$ follows from $(-)$ for Y_1 and Y_2. These together with the previous observations give that by unsplitting e the sets $A_1, A_2, \ldots, A_6$ form a C_6^*-obstacle for $p', \mathcal{P}, m'$ (where $p' = p^e$ and $m' = m^e$ are the modified functions after the unsplitting). Thus if e is the only edge of G then the proof is completed.

Assume that G contains another edge xy. Observe that $m'(P_1) = m'(P_2) = m'(P_3) = 2$ and $p' \leq 1$ again holds (if $p'(X) = 2$ for some set X then $X \subseteq V_1$ and e enters X, say $v \in X$ and $u \notin X$, but then $(\cap\cup)$ for X and X_2 and p gives a contradiction). Let $a_i \in A_i$ be the positive node in A_i (i.e. $a_1 = v_1, a_2 = v, a_3 = v_2, a_4 = v_3, a_5 = v_4, a_6 = u$). Reindex (cyclically) the sets such that $x \in A_1$. Consider the following two cases.

CASE I: If $y \notin A_1$, then we can assume that $y \in A_2$. Perform the allowed splitting-off at a_3 and a_5 and observe that we arrive in the situation given in Lemma 5.53, so there is a complete allowed splitting-off sequence (note that it is found by unsplitting two edges: uv and xy).

CASE II: If $y \in A_1$, too. Let the colour of a_1 be red. Since $A_1, A_2, \ldots, A_6$ forms a C_6^*-obstacle for $p', \mathcal{P}, m'$, there is no (a_i, xy)-switchblocking set for $i \in \{3, 4, 5\}$ (if X was such a set then $(\cap\cup)$ for X and A_1 would give that $p'(X \cup A_1) = p(A_i \cup A_1) = 1$, contradicting Definition 5.28.3). This implies by Lemma 5.19 (2d) that choosing two of a_3, a_4 and a_5 (say a_i and a_j) the one-change at a_i, x, y, a_j is admissible. There are two subcases: if one of x and y is red (say x) then the one-change at a_3, x, y, a_4 and the one at a_5, x, y, a_4 are both allowed. In the other subcase none of x and y is red: then we can assume that $c(x) \neq c(a_5)$ and $c(y) \neq c(a_3)$ (if any of these does not hold then switch x and y), so both the one-changes at a_5, x, y, a_4 and the one at a_4, x, y, a_3 are allowed. It remains to check that in both subcases one of the two allowed one-changes will result that the splitting at a_2 and a_6 becomes admissible, thus a complete allowed splitting-off sequence is found. ■

The only remaining case is when, for any edge uv of G induced in (say) V_1 the one-change neither at v_2, u, v, v_4 nor at v_2, v, u, v_4 is admissible. In other words, for every such edge there are sets X_2, X_4, such that (after possibly exchanging u and v) X_i is a (v_i, u, v)-switchblocking set for both $i = 2$ and 4. By Lemma 5.19 (2b), $X_2 \cap V_1 = X_4 \cap V_1$, so let $X_{uv} = X_2 \cap V_1$. Furthermore $X_i = X_{uv} \cup V_i$ for both $i = 2, 4$. The set X_e can be defined the same way for any e induced in some other V_i. In what follows we will restrict ourselves to the edges of G induced in V_1. Observe that $p(V_2 \cup V_4 \cup X_e) = 0$ for an edge $e \subseteq V_1$.

Lemma 5.57. *If e and f are two edges of G induced in V_1 then one of X_e and X_f contains the other.*

Proof. They cannot cross each other by the same proof as that of Claim 5.22 (choose $v_r = v_2$ and $v_s = v_4$ in that proof). Suppose that they are disjoint. Observe that $p(V_2 \cup V_4 \cup X_e \cup X_f) = 0$ (apply $(\cap\cup)$ for the crossing sets $V_2 \cup V_4 \cup X_e$ and $V_2 \cup V_4 \cup X_f$) and $p(V_3 \cup V_4 \cup X_e) = 0$ (apply $(\cap\cup)$ for the crossing sets $V_3 \cup V_4$ and $V_4 \cup X_e$). Apply $(-)$ for these two sets to get that $p(V_2 \cup X_f) = 1$, a contradiction. ∎

So the edges of G induced in V_1 can be "ordered": $E(G[V_1]) = \{e_1, e_2 \ldots, e_l\}$ such that $1 \leq i < j \leq l$ implies $X_{e_i} \subsetneq X_{e_j}$. For all $i \in \{1, 2, \ldots, l\}$ let $B_{e_i} = X_{e_{i+1}} - X_{e_i}$ (where $X_{e_{l+1}} = V_1$ here!). The sets B_e can be similarly defined for edges e induced in other sets V_i $(i \geq 2)$. For all $i = 1, 2, 3, 4$ let us define $A_i = V_i - \bigcup_{e \subseteq V_i} B_e$ (e.g. $A_1 = X_{e_1}$). Figure 5.4.2 is an illustration.

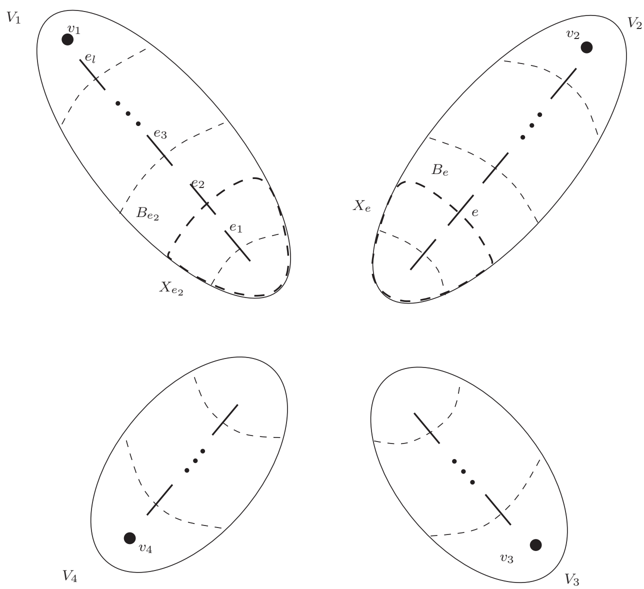

Figure 5.3: An illustration for the second subcase of CASE C

The next lemma finishes the proof of the theorem.

Lemma 5.58. *The partition* $\{A_1, A_2, A_3, A_4\} \cup \{B_e : e \in E(G)\}$ *is a* C_5^**-semiobstacle.*

Proof. We need to check that the conditions of Definition 5.27 (with, of course set function p_0 and degree-specification m_0) hold. First note that $p_0(V_{i-1} \cup A_i) = p_0(V_{i+1} \cup A_i) = 1$ for all $i = 1, 2, 3, 4$. Furthermore we will use the following observations.

Claim 5.59. *For every edge* $e \subseteq V_i$ *we have* $p_0(V_{i-1} \cup B_e) \geq 1$ *and* $p_0(V_{i+1} \cup B_e) \geq 1$.

Proof. Assume that $i = 1$ and $e = e_j$ for some $j \in \{1, 2, \ldots, l\}$. Apply $(-)$ for $V_4 \cup X_{e_{j+1}}$ and $V_2 \cup X_{e_j}$ to get the statement.

It is easy to see that Definition 5.27.1 holds: for example apply $(\cap\cup)$ for $V_2 \cup A_1$ and $V_4 \cup A_1$ to get that $p_0(A_1) \geq 1$, and use that $p_0(A_1) \leq m_0(A_1) = 1$. The proof of Definition 5.27.2 will be given later.

Let us prove 3 of Definition 5.27. Again, without loss of generality it suffices to show that $p_0(A_1 \cup A_2) = p_0(A_1 \cup B_e) = 1$ for all $e \in E(G)$. We will show that $p_0(V - (A_1 \cup B_e)) = 1$ (the proof of $p_0(A_1 \cup A_2) = 1$ is analogous), so let $X = V - (A_1 \cup B_e)$. Observe that it is enough to show that $p_0(X) \geq 1$, by $(\cap\cup)$ for X and $Y = V_j \cup B_e$ (where $V_j \subseteq X$ and $j \in \{2, 4\}$) give that $p_0(X) \leq p_0(X \cup B_e) = p_0(A_1) = 1$. In order to prove that $p_0(X) \geq 1$ we will use Claim 1.9: assume that $e \subseteq V_i$ (where i might even be 1) and apply Claim 1.9 for the subpartition $\mathcal{Q}$, if $i = 1$, and for the subpartition $\mathcal{Q} \cup \{A_i\}$, if $i \neq 1$, where $\mathcal{Q} = \{V_j : j \in \{2, 3, 4\} - i\} \cup \{B_f : f \subseteq V_i \cup V_1, f \neq e\}$. By checking the cases according to the value of i one can see that Claim 1.9 can indeed be applied.

To prove Definition 5.27.4 assume indirectly that $p_0(A_1 \cup A_3) \geq 1$ and apply Claim 1.8 for the function p_0 and the sets $A_1 \cup A_3$, $\{A_1 \cup B_e : e \subseteq V_1\}$ and $\{A_3 \cup B_e : e \subseteq V_3\}$: by the previous part we get that $p_0(V_1 \cup V_3) = p(V_1 \cup V_3) \geq 1$, contradicting the admissibility of v_1 and v_3.

Finally we prove Definition 5.27.2. Note that $p_0(A_1 \cup A_2 \cup A_3) \geq 1$ follows from $(\cap\cup)$ for $A_1 \cup A_2$ and $A_2 \cup A_3$, and similarly $p_0(A_3 \cup A_4 \cup A_1) \geq 1$. Apply $(\cap\cup)$ for these two sets to get that $1 + 1 \leq p_0(A_1 \cup A_2 \cup A_3) + p_0(A_3 \cup A_4 \cup A_1) \leq p_0(A_1 \cup A_3) + p_0(A_1 \cup A_2 \cup A_3 \cup A_4) \leq 0 + p_0(A_1 \cup A_2 \cup A_3 \cup A_4)$, by Definition 5.27.4. Choose an arbitrary $e \in E(G)$ and apply Claim 1.8 for sets $A_1 \cup A_2 \cup A_3 \cup A_4$ and $A_1 \cup B_f$ ($f \in E(G) - e$) to get that $2 \leq p(A_1 \cup A_2 \cup A_3 \cup A_4 \cup \bigcup_{f \in E(G)-e} B_f) = p(B_e)$. Since $p(B_e) \leq m(B_e) = 2$, Definition 5.27.2 is proved. ∎

Now it is easy to check whether the partition found is indeed a C_5^*-obstacle by checking the colours: if it is then there is no complete allowed splitting-off, otherwise one can find a complete allowed splitting-off by Lemma 5.43.

□

5.4.3 The minimum version

In this section, we are given a symmetric, positively crossing supermodular function on V and a partition $\mathcal{P} = \{P_1, \dots P_r\}$ of V. We show how to find a graph having its edges between different members of $\mathcal{P}$ and covering p with a minimum number of edges. The main concern here is to find an allowed degree specification m minimizing $m(V)$ such that $m(V)/2 \geq \dim(p) - 1$ also holds, and to avoid getting an obstacle.

First, we explain how to find an allowed degree specification m achieving the lower bound $m(V) = 2\phi$: such a degree-specification will be called an optimal extension. Then, we describe the instances for which the lower bound ϕ may not be achieved in the minimum version of Problem 5.25, that is the configurations. Finally, we prove the theorem solving this problem, see Theorem 5.66.

Extension

For $u \in V$ let X_u be a minimal tight set containing u (if u is not contained in a tight set then $X_u := V$). If the dependence on the degree specification m has to be emphasized, then we write X_u^m. The following lemma will be often used in this section.

Lemma 5.60. *Let m be an admissible degree-specification. Let u be a positive node and $u' \in X_u^m$. Then $m' := m - \chi_{\{u\}} + \chi_{\{u'\}}$ is again admissible, and $X_{u'}^{m'} = X_u^m$.*

Proof. If u is not in a tight set then there is nothing to prove. Otherwise, we know that X_u^m is the unique minimal tight set containing u, proving the first statement of the lemma. To prove the second statement observe that X_u^m is also m'-tight, thus $X_{u'}^{m'} \subseteq X_u^m$. Since a set X with $u \notin X \ni u'$ cannot be m'-tight by the admissibility of m and the definition of m', this finishes the proof. □

Note that the lemma also implies that if u, v are positive nodes with $X_u \cap X_v = \emptyset$ and $u' \in X_u$, $v' \in X_v$, then $m - \chi_{\{u\}} + \chi_{\{u'\}} - \chi_{\{v\}} + \chi_{\{v'\}}$ is also admissible.

An allowed degree-specification m satisfying the dimension condition (5.12) and achieving the lower bound $m(V) = 2\phi$ is called an **optimal extension for $(p, \mathcal{P})$**. Below, we describe how to find an optimal extension. The algorithm is formulated in a way such that any optimal extension can be its output.

Algorithm OPT_EXT
begin

INPUT: A **symmetric, positively crossing supermodular** function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ (given with a maximizing oracle) and a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_r\}$ of V.
OUTPUT: A degree-specification $m : V \to \mathbb{Z}_+$ satisfying (5.9)-(5.12) with $m(V) = 2\phi$.

1.1. Pick $m \in C(p) \cap \mathbb{Z}^V$ such that $m(V)$ is minimum.
1.2. If $m(V)$ is odd, let $m := m + \chi_{\{u\}}$ for some $u \in V$. Then $m(V) = 2\alpha_p$.
1.3. If $m(V) < 2(\dim(p) - 1)$ (which can be checked using Algorithm SYMCROS_COVER) then let $m = m + m'$ with an arbitrary $m' \in \mathbb{Z}_+^V$ satisfying $m'(V) = 2(\dim(p)-1) - m(V)$.
1.4. While $m(P_1) > \lceil \frac{m(V)}{2} \rceil$ (where (5.13) is assumed) do
1.5. If $X_u \nsubseteq P_1$ for some $u \in P_1 \cap V^+$ then let $m := m - \chi_{\{u\}} + \chi_{\{u'\}}$, for some $u' \in X_u - P_1$,
1.6. Otherwise let $m = m + m'$ with an arbitrary $m' \in \mathbb{Z}_+^V$ satisfying $m'(V) = 2m(P_1) - m(V)$ and $m'(P_1) = 0$.

end

It is clear that the algorithm above outputs an optimal extension. Note that if either of Step 1.2, 1.3 or 1.6 holds then $m(V) > SLB(p)$, therefore there is no obstacle for $(p, \mathcal{P}, m)$.

Configurations

It is possible that Algorithm OPT_EXT has found an optimal extension but we cannot find a complete allowed splitting-off, since there is an obstacle. In this section we show how we try to modify the optimal extension in order to getting rid of this obstacle, and we describe the structures (called configurations) where this problem cannot be avoided since any optimal extension contains an obstacle. First we give the definition of the configurations.

In order to simplify the definitions below, we introduce the following definitions. A pair (X_1, X_2) of disjoint sets of V is called a ***P*-pair** if $P \in \mathcal{P}$ and there exist a subpartition $\mathcal{F}_i$ of X_i such that $\sum_{X \in \mathcal{F}_i} p(X) = p(X_i)$ for $i = 1, 2$ and $\mathcal{F}_1 \cup \mathcal{F}_2$ is a subpartition of P. A subpartition $\mathcal{X}$ of V is called ***P*-subpartition** if $P \in \mathcal{P}$ and there exist a set $X' \subseteq X$ for every $X \in \mathcal{X}$ such that $p(X') = 1$ and $\bigcup_{X \in \mathcal{X}} X' \subseteq P$.

Definition 5.61. *A C_4^*-construction $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ for p is called a* ***C_4^*-configuration for $(p, \mathcal{P})$*** *if $\beta_p^i \leq \alpha_p$ for every $i = 1, 2, \ldots, r$ and there exist $P \in \mathcal{P}$ and $i_0 \in \{1, 2\}$ such that (A_{i_0}, A_{i_0+2}) is a P-pair.*

Definition 5.62. *A C_5^*-construction $\mathcal{A} = \{A_1, A_2, A_3, A_4, B_1, \ldots B_t\}$ for p is a* ***C_5^*-configuration for $(p, \mathcal{P})$*** *if $\beta_p^i \leq \alpha_p$ for every $i = 1, 2, \ldots, r$ and*

1. *Either there exist $P \in \mathcal{P}$ and $i_0 \in \{1, 2\}$ such that (A_{i_0}, A_{i_0+2}) is a P-pair and $\{B_1, \ldots B_t\}$ is a P-subpartition,*

2. *Or there exist $P', P'' \in \mathcal{P}$ and $j_0 \in \{1, \dots t\}$ such that (A_1, A_3) is a P'-pair, (A_2, A_4) is a P''-pair, and $\{B_j, j \neq j_0\}$ is both a P'- and a P''-subpartition.*

Definition 5.63. *A C_6^*-construction $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$ for p is a **C_6^*-configuration for $(\mathbf{p}, \boldsymbol{\mathcal{P}})$** if there exist 3 distinct colour classes P_1, P_2, P_3 such that (A_i, A_{i+3}) is a P_i-pair for $i = 1, 2, 3$.*

Note that $\beta_p^i \leq 2$ holds for every $i = 1, 2, \dots, r$ in a C_6^*-configuration, since otherwise $SLB(p) \geq 7$ would follow from Definition 5.63. There is a strong relation between configurations and obstacles, which is shown in the following two lemmas.

Lemma 5.64. *If a configuration exists for $(p, \mathcal{P})$, then for every optimal extension m, there exists an obstacle for $(p, \mathcal{P}, m)$.*

Proof. Let $\mathcal{A}$ be a configuration for $(p, \mathcal{P})$ and m an optimal extension for $(p, \mathcal{P})$. Since $\sum_{A \in \mathcal{A}} p(A) \leq \sum_{A \in \mathcal{A}} m(A) = m(V) = 2\phi = \sum_{A \in \mathcal{A}} p(A)$, we have $m(A) = p(A)$ for all $A \in \mathcal{A}$. This implies the lemma. □

We mention that if $\mathcal{A}$ is a C_5^*-configuration satisfying Definition 5.62.2 then this does not necessarily imply that an optimal extension will satisfy Definition 5.41.2: it is even possible that $|\mathcal{P}| = 2$!

Lemma 5.65. *If no configuration exists for $(p, \mathcal{P})$, then there is an optimal extension m such that no obstacle exists for $(p, \mathcal{P}, m)$.*

Proof. Let m be an optimal extension for $(p, \mathcal{P})$ and suppose that $\mathcal{A}$ is an obstacle for $(p, \mathcal{P}, m)$. Throughout this proof, we will very often replace m by $m - \chi_{\{u\}} + \chi_{\{u'\}}$ for $u' \in X_u$, using implicitly Lemma 5.60. In every case below we will show how to modify the optimal extension m this way in order to destroy the obstacle $\mathcal{A}$. Since $\mathcal{A}$ is the unique construction for p by Lemma 5.36, this way we make sure that no obstacle exists any more. Note that $X_u \subseteq A$ for every $A \in \mathcal{A}$ and $u \in A \cap V^+$.

In this proof we will assume that $m(P_1) \geq m(P_2 \geq \cdots \geq m(P_r)$ and we will use that the tie can be broken arbitrarily. We will say that P_1 is red and P_2 is blue. Note that if for example $\mathcal{A}$ is a C_4^*-obstacle but not a C_4^*-configuration then there exists a red positive node u such that $X_u \not\subseteq P_1$. As there are three different possible obstacles, there are three cases.

1. If $\mathcal{A}$ is a C_4^*-obstacle, then Definition 5.61 does not hold and for every maximum colour class P_i there exists a positive node $u \in P_i$ such that $X_u \not\subseteq P_i$. There are two cases.

(a) If $m(P_1) > m(P_2)$, then replace m by $m - \chi_{\{u\}} + \chi_{\{u'\}}$, for some $u' \in X_u - P_1$.

(b) If $m(P_1) = m(P_2)$, then pick $u_i \in P_i$ such that $X_{u_i} \not\subseteq P_i$, for $i = 1, 2$. Replace m by $m - \chi_{\{u_1\}} + \chi_{\{u'_1\}} - \chi_{\{u_2\}} + \chi_{\{u'_2\}}$ for some $u'_i \in X_{u_i} - P_i$, for $i = 1, 2$.

After these modifications Definition 5.40 does not hold any more, therefore $\mathcal{A}$ is not a C_4^*-obstacle anymore.

2. $\mathcal{A}$ is a C_5^*-obstacle. Note that if $m(P_1) = \frac{m(V)}{2}$ in a C_5^*-obstacle, then two opposite sets A_i, say A_1 and A_3, and every B_i have exactly one red positive node. There are four cases, depending on $m(P_1)$ and $m(P_2)$. In each case, we will just show that we may destroy both Definition 5.41.1 and Definition 5.41.2. Visually, Definition 5.41.1 corresponds to A_1, A_3 and every B_i has exactly one red positive node, and Definition 5.41.2 to B_{j_0} has no red and no blue positive node, A_1, A_3 have a red positive node, A_2 and A_4 have a blue positive node and every $B_j, j \neq j_0$ has exactly one red and one blue positive node.

(a) $m(P_1) = \frac{m(V)}{2}$ and $m(P_2) < \frac{m(V)}{2} - 1$. Since Definition 5.62.1 does not hold, there exists a red positive node u such that $X_u - P_1 \neq \emptyset$. Replace m by $m - \chi_{\{u\}} + \chi_{\{u'\}}$ with $u' \in X_u - P_1$. Then no colour class P may satisfy $m(P) = \frac{m(V)}{2}$. Suppose that now Definition 5.41.2 holds, then necessarily $u' \in P_2$ and $m(P_2) = \frac{m(V)}{2} - 1$. But either u was a positive node of A_i for some $i \in \{1, 3\}$ and now the positive node of A_i is blue whereas the positive node of A_{i+2} is still red. Thus Definition 5.41.2 is not satisfied. Or u belongs to some B_i and now B_i has no red positive node. Therefore Definition 5.41.2 does not hold.

(b) $m(P_1) = m(P_2) = \frac{m(V)}{2} - 1$, then every $B_j \neq B_{j_0}$ has exactly one red and one blue positive node, and B_{j_0} has no red and no blue positive node. Since Definition 5.62.2 does not hold, we may assume that there exists a red positive node u such that $X_u - P_1 \neq \emptyset$. Replace $m - \chi_{\{u\}} + \chi_{\{u'\}}$ with $u' \in X_u - P_1$. Then Definition 5.41.2 does not hold anymore, and of course 5.41.1 cannot hold either.

(c) $m(P_1) = \frac{m(V)}{2}$ and $m(P_2) = \frac{m(V)}{2} - 1$. Since Definition 5.62.1 does not hold, there exists a red positive node u such that $X_u - P_1 \neq \emptyset$. If there is such a node with $X_u - P_1 - P_2 \neq \emptyset$, then replace m by $m - \chi_{\{u\}} + \chi_{\{u'\}}$ with $u' \in X_u - P_1 - P_2$. Then Definition 5.41.1 does not hold, and either Definition 5.41.2 holds and this is Case (b), or Definition 5.41.2 does not hold.

Now, suppose for every red positive node, we have $X_u \subset P_1 \cup P_2$. Recall that at least one red positive node u satisfies $X_u \cap P_2 \neq \emptyset$.

If such a node is in A_1 or A_3, then replace m by $m-\chi_{\{u\}}+\chi_{\{u'\}}$ with $u' \in X_u \cap P_2$. Now P_2 is the only colour class satisfying $m(P_2) = \frac{m(V)}{2}$ but one of A_1, A_3 has a blue positive node and the other a red one, therefore neither Definition 5.41.1 nor 5.41.2 may hold.

Otherwise, let u be a red positive node in B_i such that $X_u \cap P_2 \neq \emptyset$ and replace m by $m - \chi_{\{u\}} + \chi_{\{u'\}}$ with $u' \in X_u \cap P_2$. Then Definition 5.41.1 does not hold for red. If afterwards Definition 5.41.1 holds for blue, then A_2 and A_4 have a blue positive node and every $B_j \neq B_i$ has exactly one red and one blue positive node. Necessarily, the positive node in B_i beside u' is neither blue nor red. But now since Definition 5.62.2 does not hold, there exists a positive node $v \notin B_i$ such that $X_v - P \neq \emptyset$ for some $P \in \{P_1, P_2\}$. If v belongs to some A_i then v is blue and it is identical with the case above for red.

Now $v \in B_j \neq B_i$. If v is red, then the modification we have performed was not a good one, therefore we will undo it and perform an other one: replace m by $m - \chi_{\{u'\}} + \chi_{\{u\}} - \chi_{\{v\}} + \chi_{\{v'\}}$ with $v' \in X_v - P_1$. Then B_i has no blue positive node and B_j has no red one, hence none of Definition 5.41.1 and Definition 5.41.2 may hold. Otherwise v is blue, then replace m by $m - \chi_{\{v\}} + \chi_{\{v'\}}$ with $v' \in X_v - P_2$. Now B_i has no red positive node and B_j has no blue ones, hence neither Definition 5.41.1 nor 5.41.2 holds.

(d) $m(P_1) = m(P_2) = \frac{m(V)}{2}$, then every B_i has exactly one red and one blue positive node. Since Definition 5.62.1 does not hold for P_1, there exists a red positive node u such that $X_u - P_1 \neq \emptyset$. If $X_u - P_1 - P_2 \neq \emptyset$, then replace m by $m - \chi_{\{u\}} + \chi_{\{u'\}}$ with $u' \in X_u - P_1 - P_2$ and this is Case (c). Otherwise, let u be a red positive node such that $X_u \cap P_2 \neq \emptyset$. Since Definition 5.62.1 does not hold for P_2, there exists a blue positive node v such that $X_v - P_2 \neq \emptyset$. If $X_v - P_2 - P_1 \neq \emptyset$, we dealt with this case just before (for P_1). Otherwise, replace m by $m - \chi_{\{u\}} + \chi_{\{u'\}} - \chi_{\{v\}} + \chi_{\{v'\}}$ with $u \in X_u \cap P_2$ and $v' \in X_v \cap P_1$. There are three cases.

   i. If at least one of u and v is in some A_i then we destroyed Definition 5.41.1, therefore $\mathcal{A}$ is not a C_5^*-obstacle any more.

   ii. If the case above does not happen, then we may assume that $u \in B_i \neq B_j \ni v$ because Definition 5.62.2 does not hold. Then B_i has two blue positive nodes and B_j has two red ones, hence neither Definition 5.41.1 nor 5.41.2 may hold.

3. If $\mathcal{A}$ is a C_6^*-obstacle, then there exists a colour class $P \in \mathcal{P}$ and a positive node

$u \in P$ such that $X_u \nsubseteq P$, then replace m by $m - \chi_{\{u\}} + \chi_{\{u'\}}$ for some $u' \in X_u - P$. Then $\mathcal{A}$ is not a C_6^*-obstacle any more.

□

We emphasize that if there exists an obstacle for $(p, \mathcal{P}, m)$, then the proof of Lemma 5.65 provides an algorithm to decide if there exists a configuration for $(p, \mathcal{P})$.

Theorem for the minimum version

By exploiting the relations between configurations and obstacles and by applying our splitting-off result, we may now prove the following theorem solving the minimum version of Problem 5.25. It states that the lower bound ϕ may always be achieved unless there exists a configuration.

Theorem 5.66. *Let $p : 2^V \to \mathbb{Z}_+$ be a symmetric, positively crossing supermodular set function and $\mathcal{P} = \{P_1, \ldots P_r\}$ a partition of V. Then the minimum number of edges between different members of $\mathcal{P}$ resulting in a graph that covers p is ϕ unless a configuration exist, in which case it is $\phi + 1$.*

Proof. Let $OPT(p, \mathcal{P})$ be the minimum number of edges between different members of $\mathcal{P}$ resulting in a graph that covers p. The following Lemmas prove the theorem.

Lemma 5.67. *$OPT(p, \mathcal{P}) \geq \phi$. If there exists a configuration for p, then the inequality is strict.*

Proof. We have seen that $OPT(p, \mathcal{P}) \geq \alpha$, $OPT(p, \mathcal{P}) \geq \dim(p) - 1$ and $OPT(p, \mathcal{P}) \geq \beta_i$ for $i = 1, \ldots r$.

Suppose there exists a configuration for $(p, \mathcal{P})$ and the inequality is not strict. Let F be a minimum set of edges such that (V, F) covers p and it satisfies the partition constraints, and let m be the degree specification defined by $m(v) := d_F(v)$ for every $v \in V$. By the minimality of F, m is an optimal extension for $(p, \mathcal{P})$. Since there is a configuration for $(p, \mathcal{P})$, by Lemma 5.64, there is an obstacle for $(p, \mathcal{P}, m)$. But this contradicts Lemma 5.44.

Lemma 5.68. *$OPT(p, \mathcal{P}) \leq \phi + 1$. If there exists no configuration for $(p, \mathcal{P})$, then the inequality is strict.*

Proof. If there exists no configuration for $(p, \mathcal{P})$, then by Lemma 5.65 there exists an optimal extension m for $(p, \mathcal{P})$ which contains no obstacle. Hence by Theorem 5.45 there exists a complete admissible splitting-off and the strict inequality follows. If there exists a configuration for $(p, \mathcal{P})$, let m be an optimal extension for $(p, \mathcal{P})$. By Lemma 5.64, there exists an obstacle for $(p, \mathcal{P}, m)$. Replace m by $m' := m + \chi_{\{u\}} + \chi_{\{v\}}$ for some u, v without violating $m(P) \leq \frac{m(V)}{2}$ for every $P \in \mathcal{P}$. Now, the sets containing u and v cannot be tight, therefore there is no obstacle for $(p, \mathcal{P}, m')$. By Theorem 5.45, there exists a complete allowed splitting-off and the inequality follows.

□

Algorithm for the minimum version

In this section, we describe the algorithm that, given a symmetric positively crossing supermodular set function $p : 2^V \to \mathbb{Z} \cup \{-\infty\}$ with a maximizing oracle, and a partition $\mathcal{P} = \{P_1, \dots P_r\}$ of V, finds a minimum number of edges between different members of $\mathcal{P}$ resulting in a graph that covers p. It consists of three major steps, extension, then splitting-off, and finally determining if a configuration exists.

1. Find an optimal extension m for $(p, \mathcal{P})$ by applying the algorithm of Section 5.4.3. Recall that $m(V) = 2\phi$.
2. Apply the algorithm described in the proof of Theorem 5.45 to $(p, \mathcal{P}, m)$.
   (a) If there is a complete allowed splitting-off, then we have found the desired graph having $\frac{m(V)}{2} = \phi$ edges.
   (b) Otherwise, we have found an obstacle $\mathcal{A}$ for $(p, \mathcal{P}, m)$.
3. Apply the algorithm described in the proof of Lemma 5.65 to $\mathcal{A}$.
   (a) If it finds another optimal extension m' for $(p, \mathcal{P})$ such that no obstacle exists for $(p, \mathcal{P}, m')$, then Theorem 5.45 provides a complete allowed splitting-off and thereby the desired graph with ϕ edges.
   (b) Otherwise $\mathcal{A}$ is a configuration for $(p, \mathcal{P})$. The algorithmic proof of Lemma 5.68 provides the desired graph with $\phi + 1$ edges.

5.4.4 Application: Partition constrained global edge-connectivity augmentation of a hypergraph

In this subsection we specialize the results of this section for the problem of **global edge-connectivity augmentation of a hypergraph with a multipartite graph**. The results of this subsection were presented at the European Conference on Combinatorics, Graph Theory and Applications (Eurocomb 2009) which was held in Bordeaux, France in September 2009: see [9].

Problem 5.69 (Partition constrained global edge-connectivity augmentation of a hypergraph)**.** *Assume that we are given a hypergraph $H_0 = (V, \mathcal{E}_0)$, a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_r\}$ of the set V and a positive integer k. The problem is to find a graph G such that $H_0 + G$ is k-edge-connected, and G contains* ***only edges between the classes*** *of $\mathcal{P}$. In the* ***minimum version*** *the number of edges of G is to be minimized. In the* ***degree-specified version*** *G has to satisfy a given degree-specification $m \in \mathbb{Z}_+^V$.*

This problem was solved by Ben Cosh [15] in the bipartition constrained case, i.e. when $r = 2$. Here we give a complete solution of this problem. Let p be the function defined by (2.1). Let $\mathbf{\Phi}$ be the maximum of the following values.

$$\begin{aligned}\alpha &= \max\{\lceil \frac{1}{2} \sum_{X \in \mathcal{X}} (k - d_{H_0}(X)) \rceil : \mathcal{X} \in \mathcal{S}(V)\},\\ \beta &= \max\{\max\{\sum_{Y \in \mathcal{Y}} (k - d_{H_0}(Y)) : \mathcal{Y} \in \mathcal{S}(P)\} : P \in \mathcal{P}\},\\ \omega(H_0) &= \max\{\#\text{component}(H_0 - \mathcal{F}) - 1 : \mathcal{F} \subseteq \mathcal{E}, |\mathcal{F}| = k - 1\}.\end{aligned}$$

One can formulate the definition of constructions, obstacles and configurations by specializing the abstract definitions given above. Interestingly, there is no C_5^*-construction for this function p used here.

Lemma 5.70. *There is no C_5^*-construction for the function defined by* (2.1).

Proof. Assume that $\mathcal{A} = \{A_1, A_2, A_3, A_4, B_1, \ldots, B_t\}$ is a C_5^*-construction. By Claim 5.31, $p(B) = 2, p(A_i \cup B) = 1, p(A_1 \cup A_2 \cup B) = 1, p(A_1 \cup A_3 \cup B) \leq 0$. Apply (1.3) and (1.4) to get that, for any crossing pair $X, Y \subseteq V$

$$p(X) + p(Y) = p(X \cap Y) + p(X \cup Y) - 2d_1(X, Y) - d_2(X, Y), \tag{5.14}$$

$$p(X) + p(Y) = p(X - Y) + p(Y - X) - 2d_1(X, V - Y) - d_2(X, V - Y). \tag{5.15}$$

(The definition of $d_1(X, Y)$ and $d_2(X, Y)$ can be found after Equation (1.3).) Equation (5.14) applied to $X = A_1 \cup B$ and $Y = A_2 \cup B$ gives that there exists exactly one hyperedge e in the hypergraph H_0 that enters A_1, A_2 and exactly one of B and $A_3 \cup A_4$. If e enters B then apply (5.15) to $X = A_1 \cup B$ and $Y = A_3 \cup B$ to get a contradiction. Otherwise, without loss of generality assume that e enters A_3: now (5.14) applied to the pair $X = A_1 \cup B$ and $Y = A_3 \cup B$ gives a contradiction. □

We specialize the definition of the **constructions** below, from which the definition of obstacles and configurations is easy to devise. To clarify the distinction from the abstract problem solved in Section 5.4, we will use the notation $\mathcal{C}_4$- and $\mathcal{C}_6$-construction. Similarly, we will use $\mathcal{C}_4$- and $\mathcal{C}_6$-obstacle/configuration for this special case.

Definition 5.71. *A partition $\{A_1, \ldots, A_4\}$ of V is a* ***$\mathcal{C}_4$-construction*** *for (H_0, k) if*

1. *There exists a set A of hyperedges such that $k - |A|$ is odd and $A = \Delta_{H_0}(A_1) \cap \Delta_{H_0}(A_3) = \Delta_{H_0}(A_2) \cap \Delta_{H_0}(A_4)$,*
2. *$\alpha = k - d_{H_0}(A_1) + k - d_{H_0}(A_3) = k - d_{H_0}(A_2) + k - d_{H_0}(A_4)$.*

Definition 5.72. *A partition $\{A_1, \ldots, A_6\}$ of V is a* ***$\mathcal{C}_6$-construction*** *for (H_0, k) if*

1. *$k - d_{H_0}(A_i) = 1$, for all $1 \leq i \leq 6$,*
2. *$k - d_{H_0}(A_i \cup A_{i+1}) = 1$, for all $1 \leq i \leq 6$, $(A_7 = A_1)$*
3. *$\alpha = 3$.*

Thus the solution of the degree-specified version of Problem 5.69 is the following.

Theorem 5.73. *We are given a hypergraph $H_0 = (V, \mathcal{E}_0)$, a partition $\mathcal{P} = \{P_1, P_2, \ldots, P_r\}$ of the set V and a positive integer k. Assume that the degree-specification $m : V \to \mathbb{Z}$ satisfies that $m(V)$ is even, $m(P_i) \leq m(V - P_i)$ for every $i = 1, 2, \ldots, r$, and $m(X) \geq k - d_{H_0}(X)$ for any nonempty $X \subsetneq V$. Then there exist a graph G satisfying the degree-specification m such that $H_0 + G$ is k-edge-connected, and G contains* ***only edges between the classes*** *of $\mathcal{P}$, unless there exists a $\mathcal{C}_4$- or a $\mathcal{C}_6$-obstacle for $H_0, k, \mathcal{P}, m$.*

The solution of the minimum version of Problem 5.69 is the following.

Theorem 5.74. *Let $H_0 = (V, \mathcal{E}_0)$ be a hypergraph, $\mathcal{P}$ a partition of V and k an integer. Then the minimum number of new edges between different members of $\mathcal{P}$ to be added to H_0 in order to make it k-edge-connected is $\Phi + 1$ if H_0 contains a $\mathcal{C}_4$- or a $\mathcal{C}_6$-configuration, and Φ otherwise.*

Chapter 6

Source location in hypergraphs – another way of edge-connectivity augmentation

In this chapter we give a different approach to edge-connectivity augmentation. This is the following: given a (hyper)graph, find a nonempty subset S of the nodes (called *source set*) such that contracting the set S will result in a (hyper)graph satisfying the edge-connectivity requirements. The objective is to minimize the total weight of the source set to be found (where nodes have nonnegative weights). This class of problems is called **source location problems**, because it comes from the following application: given a (hyper)graph $H = (V, \mathcal{E})$, decide where to locate the set of sources $S \subseteq V$ such that nodes have good edge-connectivity to this source set, and location costs are minimized. Source location problems have recently been intensively studied, not only in undirected structures but also in directed ones (e.g. digraphs). In this thesis we will concentrate on generalizations of the following problem.

Problem 6.1 (Source Location in Graphs)**.** *Let us be given a graph $G = (V, E)$, a weight function $w : V \to \mathbb{R}_+$ and a requirement function $r : V \to \mathbb{R}_+$ on the nodes. Find a nonempty subset of nodes S such that $\lambda_G(S, v) \geq r(v)$ for every node $v \in V - S$ and $w(S) := \sum_{s \in S} w(s)$ is minimum.*

This problem was raised in [1] where they show the NP-completeness of this problem. However, it is shown in [1] that the special case when w is constant can be solved in $O(|V|M)$ time, where M is the time needed for one maximum flow computation in our network. A minor observation shows that the same algorithm solves another special case, namely when r is constant, but the authors of [1] give another algorithm for this case that improves the running time to $O(|V|(|E|+|V|\log|V|))$.

In this section we introduce a generalization of Problem 6.1 for (undirected) hypergraphs. Then we generalize the problem further and formulate an abstract source location problem. Of course we can not solve this problem in general (it contains an NP-complete problem), but we observe that it can be solved with an adaptation of the greedy algorithm introduced in [1] in the special case when the requirement and the weight functions are **compatible** (see the definition in Section 6.2). This contains the case when either of these functions is uniform (constant, in other words), but in the subsequent subsection we give another algorithm for uniform requirement function that gives a better running time. In every case we also specialize our results for the hypergraphic source location problem, too. The results presented in this chapter appeared in [6].

6.1 Problem formulation

A straightforward generalization of Problem 6.1 introduced above is the following:

Problem 6.2 (Source Location in Hypergraphs). *Let us be given an undirected hypergraph $H=(V,\mathcal{E})$, a weight function $w: V \to \mathbb{R}_+$ and a requirement function $r: V \to \mathbb{R}_+$ on the nodes. Find a nonempty subset of nodes S such that $\lambda_H(S,v) \geq r(v)$ for every node $v \in V-S$, and $w(S) := \sum_{s\in S} w(s)$ is minimum.*

By Theorem 1.2 we can reformulate the problem in the following way: find a subset of nodes S such that $d_H(X) \geq \max\{r(v) : v \in X\}$ for every $\emptyset \neq X \subseteq V-S$ and $w(S)$ is minimum. However, since Problem 6.1 is a special case of this problem, this is an NP-complete problem. In what follows we will concentrate on the two special cases of this problem that were solvable for graphs; namely the constant weight and the constant requirement case, or more precisely a common generalization of these two cases when these two functions are compatible. We will show that the algorithms given in [1] can be modified appropriately to work in this more general setting, too. What is more, we will show that they work in an even more abstract environment, too. To this end let us introduce the following, abstract form of Problem 6.2. Recall that a set function $d: 2^V \to \mathbb{R}$ is posimodular if it satisfies $d(X)+d(Y) \geq d(X-Y)+d(Y-X)$ for any $X,Y \subseteq V$.

Problem 6.3 (Abstract Source Location). *We are given a function $d: 2^V \to \mathbb{R}_+$ that is posimodular and submodular, a weight function $w: V \to \mathbb{R}_+$ and a requirement function $r: V \to \mathbb{R}_+$ on the nodes. Find a subset of nodes S with $w(S)$ minimum subject to*

$$d(X) \geq \max\{r(v) : v \in X\} \text{ for every } X \subseteq V-S. \tag{6.1}$$

For brevity, a set $S \subseteq V$ satisfying (6.1) will be called **source set**. As a remark we mention that the nonnegativity of the function d is not really a restriction: an arbitrary posimodular function d' attains its minimum at the empty set, so defining $d(X) = d'(X) - d'(\emptyset)$ gives a nonnegative posimodular function and we can modify the requirement function similarly to be able to reduce the more general problem without the nonnegativity constraints to our problem. On the other hand, the nonnegativity of the weight function is a natural requirement, since any superset of a valid source set is again a valid source set. Also, we could have required the nonemptyness of a source set: that would not have changed the problem significantly.

In the following sections we will show that the special case of this abstract problem when the requirement and the weight functions are compatible can be solved using appropriate adaptations of the algorithms given in [1].

Let us introduce some terminology. A set $X \subseteq V$ is called **deficient** if $d(X) < \max\{r(v) : v \in X\}$. It is obvious that a set S is a valid source set if and only if it meets every deficient set, which is equivalent to requiring that S has to meet every **minimal deficient set**.

6.2 Compatible requirement- and weight function

In this section we will give an algorithm to solve Problem 6.3 in the special case when the requirement- and weight functions are compatible. This is an adaptation of the greedy algorithm given in [1]. Let us define first what we mean by these functions being compatible.

Definition 6.4. *Two functions $r, w : V \to \mathbb{R}$ are **compatible** if there is an ordering $v_1, v_2, \ldots, v_n$ of V such that $r(v_1) \leq r(v_2) \leq \cdots \leq r(v_n)$ and $w(v_1) \geq w(v_2) \geq \cdots \geq w(v_n)$.*

Observe that compatibility of two functions can be easily checked in $O(n \log n)$ time and it is indeed a symmetric relation. Now we can give the greedy algorithm to solve Problem 6.3: in this algorithm we only assume the *posimodularity* of the function d but we leave one question open that will only be answered when d is also submodular!

Algorithm GREEDY
begin

> **INPUT** A posimodular function $d : 2^V \to \mathbb{R}_+$ given with a function evaluation oracle, a requirement function $r : V \to \mathbb{R}_+$ and a weight function $w : V \to \mathbb{R}_+$ on the finite set V. Assume that r and w are compatible functions.
>
> **OUTPUT** A minimum weight source set S.

1.1. Let $S = V$. Order V s.t. $r(v_1) \leq r(v_2) \leq \cdots \leq r(v_n)$ and $w(v_1) \geq w(v_2) \geq \cdots \geq w(v_n)$.

1.2. For $i = 1$ to n do
1.3. (*) If there is no deficient set X with $S \cap X = \{v_i\}$ then $S := S - v_i$.
1.4. Output S.
end

One thing remains to make clear: how to implement Step (*); to be able to do this we will also assume the submodularity of the function d. Let us speak about this later and first check the correctness of the algorithm: the following argument was taught to the author by András Frank. Let us denote the current set S in the ith iteration *before* executing step (*) by S_i (so $S_1 = V$) and let the output of the algorithm be $S_{n+1} = \{v_{i_1}, v_{i_2}, \ldots, v_{i_t}\}$ for some $t \geq 0$. A simple inductive argument shows that S_i is a valid source set for any i between 1 and $n+1$. We only need to show that S_{n+1} has minimum weight among the source sets. If, in the ith step of the for loop, $S_i - v_i$ is not a valid source set then there must be a deficient set X_i with $S_i \cap X_i = \{v_i\}$. We can assume that this X_i is minimal deficient: there is a minimal deficient set in X_i but it must contain v_i otherwise S_i was not a valid source set. In particular $X_i \subseteq \{v_1, v_2, \ldots, v_i\}$ which also implies that $\max\{r(v) : v \in X_i\} = r(v_i)$. We will show that these sets $X_{i_1}, X_{i_2}, \ldots, X_{i_t}$ are pairwise disjoint minimal deficient sets. Assume that this is not the case and suppose $i_j < i_k$ are such that $X_{i_j} \cap X_{i_k} \neq \emptyset$. It is clear that $v_{i_k} \notin X_{i_j}$ because $X_{i_j} \subseteq \{v_1, v_2, \ldots, v_{i_j}\}$. But $v_{i_j} \in X_{i_k}$ can not hold either since in the i_kth iteration X_{i_k} is only covered by v_{i_k} from among the nodes of S_{i_k} which also contains v_{i_j}. But then we have a contradiction from

$$r(v_{i_j}) + r(v_{i_k}) > d(X_{i_j}) + d(X_{i_k}) \geq d(X_{i_j} - X_{i_k}) + d(X_{i_k} - X_{i_j}) \geq r(v_{i_j}) + r(v_{i_k}).$$

(The first inequality follows from the deficiency of sets X_{i_j} and X_{i_k} and the last is because they are minimal deficient sets.) From these observations the optimality of the algorithm follows: we have given an optimal covering of the disjoint minimal deficient sets $X_{i_1}, X_{i_2}, \ldots, X_{i_t}$.

6.3 Implementation of Step (*)

The only thing to show is the subroutine that implements Step (*). We note that a subroutine checking whether a given set S is a valid source set or not would also suffice but the one we gave is easier to implement in our case (and also gives a better running time). In the ith iteration we define the set function $d_i : 2^{V-S_i} \to \mathbb{R}$ with $d_i(X) := d(X + v_i)$ for any $X \subseteq V - S_i$. Clearly, $S_i - v_i$ is a valid source set if and only if $\min\{d_i(X) : X \subseteq V - S_i\} \geq r(v_i)$. In order to be able to decide this we should be able to minimize the posimodular function d over the nonempty subsets, or in other words we should be able to minimize an intersecting posimodular function. It is an open question already mentioned in [38] whether this can be done for an arbitrary posimodular set function d. Therefore we also assume that d (and thus d_i for every i) is submodular in the rest of this section. So any algorithm for submodular function minimization will solve this problem. This yields the following result:

Theorem 6.5. *Algorithm GREEDY solves the Abstract Source Location Problem with compatible requirement and weight function in $O(n\,\mathrm{SFM}(n,\gamma))$ time.*

Here $\mathrm{SFM}(n,\gamma)$ denotes the time needed to decide whether a submodular function on an n element ground set takes on a value below a given bound, where the function is given by an oracle and γ denotes the time of a call to this oracle.

We mention that the dual solution (disjoint deficient sets) can be obtained, since algorithms for submodular function minimization can produce these, too. We also point out that deciding whether a submodular function takes a value less than a given bound B can be formulated as Problem 6.3 with uniform weight function (following the ideas of Narayanan [35], for example), so any algorithm solving Problem 6.3 with uniform weight function will need essentially at least $\mathrm{SFM}(n,\gamma)$ time.

The following corollary is a direct extension of Theorems 1 and 2 of [1] and it improves the result (Theorem 24) in [38] by a factor of n:

Corollary 6.6. *Problem 6.3 with a uniform weight function (and arbitrary requirements) can be solved in $O(n\,\mathrm{SFM}(n,\gamma))$ time.*

Another corollary of Theorem 6.5 is that Problem 6.3 with a uniform requirement function (and arbitrary weights) can be solved in $O(n\,\mathrm{SFM}(n,\gamma))$ time. However, in Section 6.4 we will improve the running time for this special case.

We can specialize our algorithm to the hypergraphic source location problem: in this case Step (*) can be regarded as a minimum $(S-v_i, v_i)$ cut problem in our hypergraph $H=(V,\mathcal{E})$ that can be translated to a maximum-flow problem in a graph that has $O(n+|\mathcal{E}|)$ nodes and $O(||\mathcal{E}||)$ edges, where $||\mathcal{E}||$ denotes the sum of the cardinalities of the hyperedges. This yields the following result:

Theorem 6.7. *Algorithm GREEDY solves the Source Location Problem in Hypergraphs with compatible requirement and weight function in $O(nM(n+|\mathcal{E}|, ||\mathcal{E}||))$ time, where $M(n',m')$ denotes the time needed for a maximum-flow computation in a graph with n' nodes and m' edges.*

6.4 Improving the running time for uniform requirements

In this section we will give an algorithm to solve Problem 6.3 in the special case when the requirement function is constant, that is $r(v) = k$, $\forall v \in V$, for some $k \in \mathbb{R}_+$. This is an adaptation of the algorithm solving a special case of this problem that was given in [1]. Let us make one simple observation: minimal deficient sets are disjoint in this case. This is proved by the following argument. Suppose X and Y are two such sets with $X \cap Y \neq \emptyset$. Since they can not contain each other, $\emptyset \neq X - Y \subsetneq X$ and $\emptyset \neq Y - X \subsetneq Y$ which gives a contradiction together with

$$k + k > d(X) + d(Y) \geq d(X - Y) + d(Y - X) \geq k + k.$$

(The first inequality follows from deficiency and the last from minimality.)

In [37] Queyranne introduces the notion of a **pendent pair** and gives an algorithm to find a pendent pair. The algorithm computes an ordering of the nodes (that will be called **Max-Adj** ordering here) and finds the pendent pair using this ordering. This Max-Adj ordering is nothing else but the appropriate generalization of the ordering used in the algorithm by Nagamochi and Ibaraki to find a minimum cut in a graph. Let us give the definitions and results.

Definition 6.8. *If $h : 2^V \to \mathbb{R}$ is a symmetric submodular function then an ordering $(v_1, v_2, \ldots, v_n)$ is a* Max-Adj *ordering for this function if it satisfies the following:*

$$h(\{v_i\}) - h(\{v_1, \ldots, v_i\}) \geq h(\{v_j\}) - h(\{v_1, \ldots, v_{i-1}, v_j\}) \ \forall 2 \leq i \leq j \leq n.$$

We note that the first element v_1 of such an ordering can be specified arbitrarily.

Definition 6.9. *If $h : 2^V \to \mathbb{R}$ is a symmetric submodular function then an ordered pair of nodes (s, t) is called a **pendent pair** if*

$$h(\{t\}) = \min\{h(X) : X \subseteq V, s \notin X, t \in X\}.$$

Lemma 6.10 ([37])**.** *If $(v_1, v_2, \ldots, v_n)$ is a* Max-Adj *ordering for the function h then (v_{n-1}, v_n) is a pendent pair. Such an ordering can be calculated with $O(n^2)$ calls to the h-value oracle and $O(n^2)$ other operations.*

After these preliminaries we can present the algorithm that solves Problem 6.3 when the requirement function is uniform. Let us first give a sketch of this algorithm.

We want to calculate a minimum weight source set S and we start with $S = \emptyset$. The main idea is the following: the minimal deficient sets form a subpartition of V. In each step of the algorithm we find a suitable pair of nodes that is not separated by a (yet uncovered) minimal deficient set. Then we contract this pair of nodes, and repeat this until one minimal deficient set becomes a singleton. Then we notice this and include an element of this set in S. But what shall we do with the minimal deficient sets that are already covered? We should somehow increase the value of function d on sets that contain such a set: this is best described using an auxiliary graph and its cut function.

After a certain number of such contractions every node in the resulting set is the image of a contracted set in the original ground set. For every such contracted set we remember the "cheapest" element in it, i.e. the one with the smallest weight. This is done with the ancestor function in the algorithm below. This way, when we contract a minimal deficient set into a singleton we can pick the cheapest element from it to include in our set S. The correct formulation of the algorithm is described by the following pseudocode.

Algorithm CONSTANT_DEMAND

begin

INPUT A posimodular and submodular function $d : 2^V \to \mathbb{R}_+$, a requirement $k \in \mathbb{R}_+$ and a weight function $w : V \to \mathbb{R}_+$ on the finite set V.

OUTPUT A minimum weight source set S.

1.1. Let $S = \emptyset$, $V' = V + s$ with a new node s and $G = (V', E)$ an auxiliary graph: in the beginning E is empty but later we add edges in E (one endpoint of these will always be s). Define the symmetric submodular function $h : 2^{V'} \to \mathbb{R}$ by $h(X) = d(X) + kd_G(X)$ for any $X \subseteq V$ and $h(X) = h(V' - X)$ if $s \in X \subseteq V'$.

1.2. For each $v \in V$

1.3. $\quad$ $\text{ancestor}(v) := v$.

1.4. $\quad$ If $d(\{v\}) < k$ then $S := S + v$ and add an edge between s and v to G (so the function h changes here!).

1.5. End for.

1.6. While $|V'| > 2$ and $\exists v \in V' - s$ with $(s, v) \notin E$

1.7. $\quad$ Construct the MAX-ADJ ordering of V' for the function h starting with s and take the last two elements in this ordering (a pendent pair) $(v_{n'-1}, v_{n'})$.

1.8. $\quad$ Contract $v_{n'-1}$ and $v_{n'}$ into a node v' and let the ancestor of v' be the cheaper of the ancestor of $v_{n'-1}$ and that of $v_{n'}$ (so V' gets smaller: the function h and the graph G follow these changes the obvious way).

1.9. $\quad$ If $h(\{v'\}) < k$ then add an edge between s and v' to G and $S := S + \text{ancestor}(v')$ (function h changes, too!).

1.10. End while.

1.11. Output S.

end

Let us prove the correctness of the above algorithm. In the argumentation below we will think of the set V as the original ground set, while the set V' always means the current ground set after some contractions (note that s is never contracted with any other node). Also, for a set $X \subseteq V$ we will use the notation X' for the image of this set after the contractions made so far: this only makes sense if there is no set $Y \subseteq V$ with $X \cap Y \neq \emptyset$ and $(V - X) \cap Y \neq \emptyset$ that was contracted.

Lemma 6.11. *In Step 1.7 of the above algorithm*

(i) if $X \subseteq V$ is a minimal deficient set that is not covered yet (i.e. $S \cap X = \emptyset$ with the current S) then the image X' of X makes sense, and

(ii) there is no (yet uncovered) minimal deficient set $X \subseteq V$ for which $|X' \cap \{v_{n'-1}, v_{n'}\}| = 1$.

Proof. Initially (i) is true. At any stage, if (i) is true, then the statement of (ii) makes sense: assume it is not true and there is a minimal deficient set $X \subseteq V$ for which X' separates $v_{n'-1}$ and $v_{n'}$. But then $d(X') = h(X')$ since X is not covered yet by S and

$$k > d(X) = d(X') = h(X') \geq h(v_{n'}) \geq k,$$

gives a contradiction (the second inequality follows from the properties of the Max-Adj ordering; for the last inequality observe that we always assure in the algorithm that $h(v) \geq k$ for singletons $v \in V' - s$). Finally, if (ii) is true then (i) will be true in the next iteration, after contracting $v_{n'-1}$ and $v_{n'}$, too. □

Lemma 6.12. *Algorithm CONSTANT_DEMAND solves Problem 6.3 with uniform requirement function in $O(n^3\gamma + n^3)$ time (where γ denotes the time of a call to the d-value oracle).*

Proof. The correctness of the algorithm follows from Lemma 6.11 and the definition of the ancestor function. It is straightforward to check that Step 1.7 dominates the algorithm which gives the running time indicated above. □

If we specialize our algorithm to hypergraphs, that is to Problem 6.2, then we can substitute Step 1.7 of our algorithm with a similar subroutine described in [32]. This yields the following running time:

Lemma 6.13. *Algorithm CONSTANT_DEMAND solves Problem 6.2 with uniform requirement function in $O(n^2 \log(n) + n||\mathcal{E}||)$ time ($||\mathcal{E}||$ denotes the sum of the cardinalities of the hyperedges).*

Chapter 7

Open problems

I don't think that anyone ever has finished a thesis (ar any mathematical writeup) without leaving some of the questions he or she wanted to answer open. Let me enumerate the problems that I was wondering about during writing this thesis.

1. **Maximizing skew-supermodular functions, maximizing crossing (intersecting) negamodular functions**. In Section 1.4 we have observed that a maximizing oracle can be implemented for a crossing supermodular function $p : 2^V \to \mathbb{R}$ given with an evaluation oracle by using standard submodular function minimization techniques. However this is unknown if the function p is only skew-supermodular. The simplest open problem is whether a maximizing oracle can be implemented for an intersecting negamodular function $p : 2^V \to \mathbb{R}$ given with an evaluation oracle. Note that these problems were almost always present in our abstract algorithms, even in Section 6.2, where we wanted to minimize an intersecting posimodular function.

2. **Rank-respecting augmentation of a hypergraph with negamodular constraints for $\gamma = 2$.** This problem is a slight generalization of the *rank-respecting node-to-area connectivity augmentation problem in hypergraphs*. In Section 4.4.3 we have solved this problem for any $\gamma \geq 3$ but we did not bother about the $\gamma = 2$ case, as the interesting special case (the node-to-area connectivity augmentation in graphs) has already been solved by Ishii and Hagiwara.

3. **Rank-respecting global arc-connectivity augmentation of mixed hypergraphs.** In Section 4.4.2 we have shown that the Algorithm GREEDYCOVER applied to the global arc-connectivity augmentation of a mixed hypergraph M finds graph edges plus a hyperedge that is at most one bigger than the rank of M. Probably, similarly to the results in Section 4.4.3 we could solve the rank-respecting version of this problem, too.

4. **Covering a (positively) skew-supermodular function of form $p_0 - d_G$ with graph edges, where p_0 does not take the value 1.** In the proofs of the NP-completeness of Problem 4.1 one seems to need that the skew-supermodular function often takes the value 1. Possibly this is a key point, as was the case with the node-to-area connectivity augmentation in graphs, and we possibly obtain a polynomially solvable problem if we restrict ourselves to functions of the above form.

5. **Covering a crossing supermodular function with graph edges.** Benczúr and Frank [5] has solved the problem of *covering a symmetric crossing supermodular function with graph edges*. Probably this could be generalized to not necessarily symmetric functions, too. An application would be the *global arc-connectivity augmentation of mixed hypergraphs with graph edges*.

6. **Minimum node-cost partition constrained covering of a a symmetric positively crossing supermodular function (with graph edges).** In Section 5.4 we have solved the minimum- and the degree-specified version of Problem 5.25. The minimum node-cost version does not follow straightforwardly from the degree-specified version as usually, because of the partition constraints. However, probably this is polynomially solvable, too.

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Összefoglaló

A dolgozat nagyobb részében azzal a hagyományos élösszefüggőség-növelési feladattal foglalkozunk, amikor új élek illetve hiperélek hozzáadásával kell egy gráf illetve hipergráf élösszefüggőségét megnövelni. Hangsúlyozzuk, hogy míg a növelendő hipergráf tartalmazhat irányított hiperéleket is, addig a növelő (hiper)gráf mindig irányítatlan ebben a dolgozatban. A legfontosabb célfüggvény mindig a hozzáadandó új (hiper)élek összméretének minimalizálása. Ezeket a feladatokat mindig egy ferdén szupermoduláris halmazfüggvény gráfélekkel avagy hiperélekkel való fedésének absztrakt feladatává fogalmazzuk át. Alkalmazásként azonban a következő élösszefüggőség-növelési feladatok variánsai lebegnek a szemünk előtt: hipergráfok globális illetve lokális élösszefüggőségének növelése, vegyes hipergráfok globális élösszefüggőségének növelése, illetve az úgynevezett node-to-area élösszefüggőség-növelési feladat. A variánsok főleg abban különböznek, hogy megengedjük e tetszőlegesen nagy hiperélek hozzáadását, avagy megpróbáljuk ezek méretét korlátozni, extrém esetben csak gráféleket engedünk meg.

A megfelelő bevezetések után a dolgozat 3. fejezetében azt az esetet vizsgáljuk, amikor tetszőleges méretű hiperélek hozzáadása megengedett, és Szigeti Zoltán ide vonatkozó tételét általánosítjuk több irányban is. A 4. fejezetben ezzel éppen ellentétben azt vizsgáljuk, hogy mi történik, ha az ember csak gráfélek hozzáadásával próbálkozik. Ez a megközelítés megválaszolja azt az esetet, amikor az új hiperélektől azt várjuk, hogy a rangot ne (nagyon) növeljék meg. Végezetül az 5. fejezetben azt vizsgáljuk, hogy mi van, ha ragaszkodunk ahhoz, hogy csak gráféleket engedünk meg, de cserébe csak a hipergráfok globális élösszefüggőségének növelésére korlátozódunk. A feladat absztrakt alakjának megoldását Benczúr és Frank tétele szolgáltatja: erre adunk egy viszonylag egyszerű bizonyítást, ami lehetővé teszi, hogy a feladat partíciókorlátos változatát is megoldjuk.

A dolgozat 6. fejezetében egy másik élösszefüggőség-növelési fogalmat tekintünk, az úgynevezett forrástelepítési feladatkört. Itt a (hiper)gráf élösszefüggőségét egy alkalmasan választott ponthalmaz összehúzásával akarjuk megnövelni. Általánosítjuk Arata és társai gráfokra vonatkozó eredményeit egy absztrakt formában, ami a hipergrafikus változatot is magában foglalja.

Summary

The main part of this thesis deals with the traditional notion of edge-connectivity augmentation, when the edge-connectivity of a graph or hypergraph has to be augmented by introducing new edges or hyperedges. We emphasize that though the hypergraph to be augmented can even contain directed hyperedges, the new (hyper)edges are always undirected in this thesis. The most important objective function is always to minimize the total size of the new (hyper)edges. The problems are always treated in the abstract framework of covering a skew-supermodular function with graph edges or hyperedges. However, as applications we will consider variants of the following edge-connectivity augmentation problems: global or local edge-connectivity augmentation of hypergraphs, global arc-connectivity augmentation of mixed hypergraphs, and the so called node-to-area edge-connectivity augmentation problem. The variants mainly differ on whether we allow the addition of arbitrarily large hyperedges, or we restrict the size of these, in the extremal case we only allow graph edges.

After the appropriate introduction, Chapter 3 deals with the case when we allow hyperedges of arbitrary size: we generalize the theorem about this case due to Zoltán Szigeti in many directions. On the other hand in the 4th chapter we try to answer what happens if one wants to use only graph edges. This approach answers the problem when we want that the rank of the hypergraph should not increase (too much). Finally, in Chapter 5 we insist on using only graph edges, but we restrict ourselves to the global edge-connectivity augmentation of undirected hypergraphs. The abstract form of this problem was solved by Benczúr and Frank: we give a relatively simple proof of their result which enables us to generalize it to the partition constrained case, too.

In Chapter 6 we investigate a different edge-connectivity augmentation technique, the class of the so called source location problems. Here the edge-connectivity of the (hyper)graph is to be augmented by contracting a suitable set of nodes. We generalize the results of Arata et al on graphs to an abstract framework which also includes the hypergraphic version of the problem.